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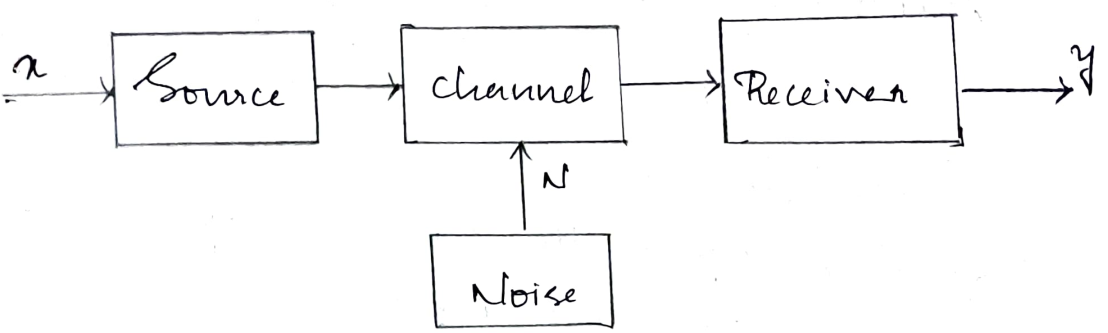
Information theory deals with mathematical modeling and analysis of a communication system rather than physical sources and physical channel.

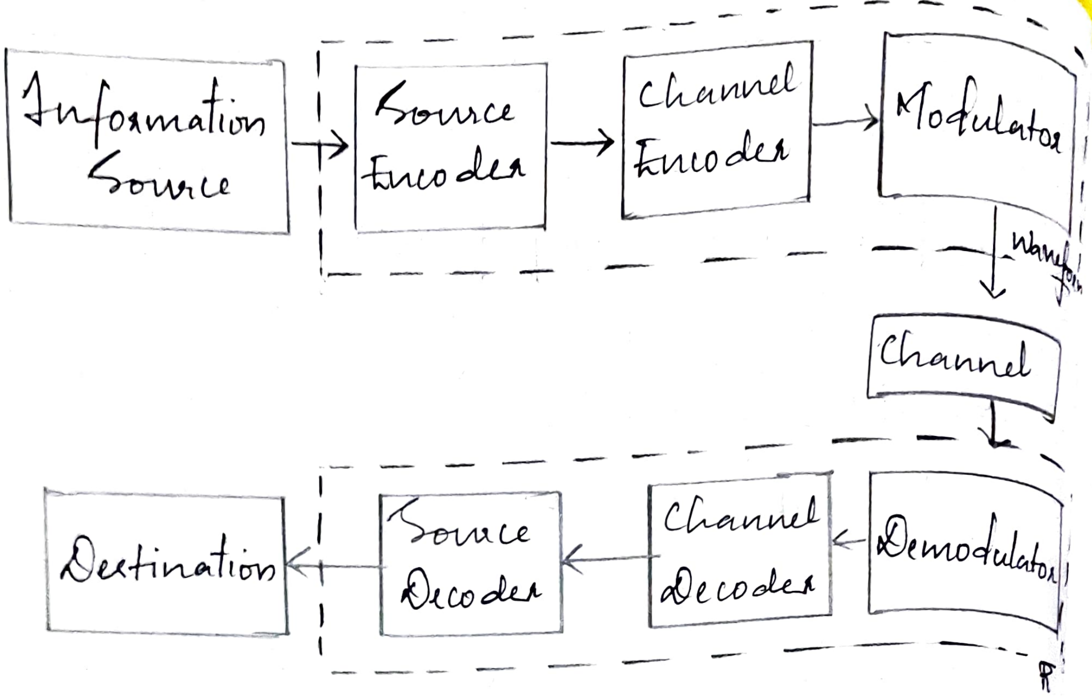


The transmitted signal is x. The noise signal is N. Therefore, the received signal $y = x + N$. The sources $S_0, S_1, S_2, \ldots, S_n$.

Two important factors of a communication system are:

1. Distorting effect of channel imperfection and noise should be minimised.

2. The number of messages send over the channel in a given time [Transmission rate] should be maximised.



* <u>Source Encoder</u>:

→ It transform the Source output into a Sequence of binary bit.

→ It also remove redundant information from message Signal.

→ It is responsible for the efficient use of the channel.

→ The resulting Sequence of Symbol is called Source code word.

* <u>Channel Encoder</u> [Error Correction & detection]

→ The Source code word Send as a data Stream is processed to the Channel encoder which provides a new Sequence of Symbol is called channel code word.

→ The channel code word are longer than the Source code word because the redundant bits are build in the channel code.

* **Modulator:**

→ The discrete Symbols are not suitable for transmission over the physical channel, the modulator represent each symbol of the channel code word by a corresponding analog signal approximately selected from the finite set of possible analog signal.

→ The sequence of analog symbol produced by the modulator is called a waveform which is suitable to transmission over the channel.

* Typically the transmission channel include telephone line, high frequency radio link, microwave and satellite link etc.

* On a telephone link the disturbance may come from switching the impulse noise, thermal noise or cross talks from other lines. ~~The~~

* The demodulator process each received waveform and produces and o/p. The sequence of demodulated o/p corresponded to the encoded sequence is called received signal.

* The channel decoder transforms the received sequence into a binary sequence called as an estimat~~edy~~ signal.

* The decoding strategy is based on the roots of channel encoding and noise characteristics of the channel.

* The <u>Source decoder</u> transform the estimated Sequence into an estimate of the Source o/p and delivers it to the destination and it involves analog to digital conversion.

* In a well designed System the estimate Will be the reproduction of the Source o/p at its maximum except when the channel is Very noisy.

Information is inversly proportional to probability. ie; $I \propto \frac{1}{P}$

$\Rightarrow \quad I = \log\left(\frac{1}{P}\right)$

As the probability increases information decreases. S_K is denoted as the Source of information ie; $S_K = 1/P_K$

$$\boxed{I_K = \log_2\left(\frac{1}{P_K}\right)}$$

If $P_K = 1 \Rightarrow I_K = 0$

If $P_K = 0 \Rightarrow I_K =$ Infinity.

$\Rightarrow \log_y(x) = \dfrac{\log_2(x)}{\log_2(y)}$

$\Rightarrow \log xy = \log x + \log y$

$\Rightarrow \log \frac{a}{b} = \log a - \log b$

$\Rightarrow \log x^m = m \log x$

$\Rightarrow \log \frac{1}{a} = -\log a$

Q. A discrete memoryless Source have 8 Symbol with equal probability. Calculate the Self information of a Symbol in the Symbol Set.

Sol: $I_k = \log_2 \left[\frac{1}{P_k}\right]$

No. of Symbols = 8

∴ The probability of Symbols, $P_k = \frac{1}{8}$

$I_k = \log_2 \left(\frac{1}{1/8}\right) = \log_2 8$

$I_k = \underline{3 \text{ bits}}$.

Entropy:

Average information content of channel is called as an entropy. It is denoted as H [S] or H [Y] or marginal entropy i.e;

$H[S] = \sum_{k=1}^{n} P_k I_k$ Unit: bits/symbol.

$$\boxed{H[S] = \sum_{k=1}^{n} P_k \log_2 \left(\frac{1}{P_k}\right)}$$

Q. The probability of a Signal of a discrete memoryless Source is given as (0.1, 0.3, 0.5, 0.05, 0.005). Find the entropy of the Source.

Sol: $H(S) = \sum_{k=1}^{n} P_k \log_2 \left(\frac{1}{P_k}\right)$

$= 0.1 \log_2 \left(\frac{1}{0.1}\right) + 0.3 \log_2 \left(\frac{1}{0.3}\right) + 0.5 \log_2 \left(\frac{1}{0.5}\right) + 0.05 \log_2 \left(\frac{1}{0.05}\right) + 0.005 \log_2 \left(\frac{1}{0.005}\right)$

$= \underline{1.075 \text{ bits/Symbol}}$.

Q. Calculate the entropy of a Source having equal probability Symbol provided that total number of Symbol is 16.

Sol: $$H[S] = \sum_{K=1}^{n} P_k \log_2\left(\frac{1}{P_K}\right)$$

$$S_K = 16 \qquad P_K = \frac{1}{16}$$

$$\therefore \cancel{H[S]} = \frac{1}{16} \log_2\left(\frac{1}{1/16}\right)$$

$$= \underline{\underline{0.25}}$$

$$\therefore H[S] = 16 \times 0.25$$

$$= \underline{\underline{4 \text{ bits/Symbol}}}$$

imp <u>Entropy of a binary memoryless Source.</u>

Two Symbols, $S = [0, 1]$ with the probability $P = [P, 1-P]$ [Entropy of two different Source]

$$H[S] = \sum_{K=1}^{n} P_K \log_2\left(\frac{1}{P_K}\right)$$

$$H[P] \,/\, H[S] = \sum_{K=1}^{n} P \log_2\left(\frac{1}{P}\right) + (1-P) \log_2\left(\frac{1}{1-P}\right)$$

$$H[P] = P \log_2\left(\frac{1}{P}\right) + (1-P) \log_2\left(\frac{1}{1-P}\right)$$

<u>Case 1:</u>

$$P = 0 \; ; \; 1-P = 1$$

$$H[P] = \underline{\underline{0}}$$

Case 2:

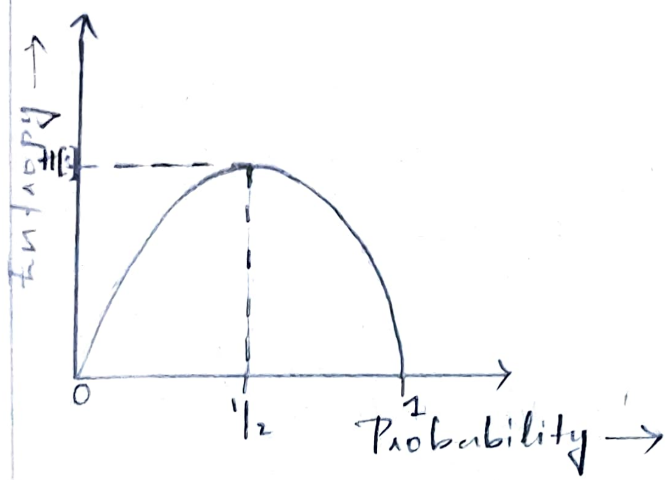
$P = 1, \; 1-P = 0$

$H[P] = 0$

Case 3:

$P = 1/2$ and $(1-P) = 1/2$

$H[P] = 1.$



1/1/24 **Information rate [R]:**

total amount of information

Unit = bits/symbol.

$$H[S] = \sum_{k=1}^{n} P_k \log_2\left(\frac{1}{P_k}\right)$$

The symbol rate is denoted as r and

r = no. of symbol/sec $= 1/T$

$$R = r \times H[S]$$

Q A discrete memoryless source emit symbol at each one milli second with the probability 0.4, 0.3, 0.2, 0.1. Calculate the information rate.

Sol: $R = r H[S]$

$r = 1/10^{-3} = 10^3$ bits/symbol.

$$H[S] = \sum_{k=1}^{n} P_k \log_2\left(\frac{1}{P_k}\right)$$

$$= 0.4 \log_2 \frac{1}{0.4} + 0.3 \log_2 \frac{1}{0.3} + 0.2 \log_2 \frac{1}{0.2} + 0.1 \log_2 \frac{1}{0.1}$$

$= 1.846$ bits/Symbol.

$\therefore$ Information rate, $R = 10^3 \times 1.846$

$= 1846$ bits/Symbol

Q An analog Signal bandlimited to B Hz is Sampled at nyquist rate the Samples are quantised into 4 level. The quantisation level are assumed to be independent and occured with the probability $P_1 = P_4 = 1/8$ $P_2 = P_3 = 3/8$ find the information rate of the Source? [Assume B to be 100 Hz if not given]

Sol: $B = 100 Hz$. $\boxed{f_m = B}$

Nyquist rate $r = 2 f_m$

$= 2 \times 100$

$= 200$

$$H[S] = \sum_{K=1}^{n} P_K \log_2\left(\frac{1}{P_K}\right)$$

Given, $P_1 = P_4 = 1/8$; $P_2 = P_3 = 3/8$

$$\therefore H[S] = 2 \times \frac{1}{8}\log_2(8) + 2 \times \frac{3}{8}\log_2(8/3)$$

$$= 2 \times \frac{1}{8} \times 3 + 2 \times \frac{3}{8} \times 1.415$$

$$= \frac{3}{4} + 1.06127$$

$= 1.8112$ bits/Symbol

Information rate $= r H[S]$

$= 200 \times 1.8112$

$R = 362.24$ bits/Symbol

Q. A Telegram Source hase two symbol dot and dash. The dot duration is 0.6 Second and the dash duration is half the dot duration. The probability of dot occurance is thrice that of dash and the time between the Symbol is 0.1 Second. Calculate the information rate of the Source.

Sol: $T_{dot} = 0.6$ Second.

$$T_{dash} = \frac{1}{2} T_{dot} = \text{half } T_{dot}$$

$$= 0.3 \text{ Sec}$$

$$P_{dot} = 3 \times P_{dash} \qquad T_{gap} = 0.1 \text{ Second}$$

$$P_{dot} + P_{dash} = 1$$

$$3P_{dash} + P_{dash} = 1$$

$$4P_{dash} = 1$$

$$P_{dash} = \frac{1}{4}$$

$$\therefore P_{dot} = \frac{3}{4}$$

Information rate $R = r H(s)$

$$r = \frac{1}{T_{avg}} \qquad T_{avg} = T_{dot} \times P_{dot} + T_{dash} P_{dash} + T_{gap}$$

$$\therefore T_{avg} = 0.6\left(\frac{3}{4}\right) + 0.3\left(\frac{1}{4}\right) + 0.1$$

$$= 0.625$$

$$\therefore r = \frac{1}{0.625}$$

$$= 1.6.$$

$$H[S] = \sum_{k=1}^{n} P_k \log_2 \frac{1}{P_k}$$

$$= \frac{3}{4} \log_2\left(1/(3/4)\right) + \frac{1}{4} \log_2\left(1/(1/4)\right)$$

$$= 0.31127 + \cancel{0.07525}\ 0.5$$

$$= 0.811 \text{ bits/Symbol.}$$

$R = r\,H(S)$
$= 1.6 \times 0.811$
$= 1.298$ bits/Symbol.

Q Given a binary source of two symbols x_1 and x_2. x_2 is ~~..~~ twice as long as x_1 and half probable duration of x_1 is 0.35. Calculate the information rate.

Sol: $x_2 = 2x_1$

time duration $T_{x_2} = 2T_{x_1}$

$\Rightarrow T_{x_1} = 0.35$

$T_{x_2} = 2 \times 0.35 = 0.7$

$P_{x_2} = \text{half } P_{x_1}$

$P_{x_2} + P_{x_1} = 1$

$\therefore P_{x_2} = \frac{2}{6} = \frac{1}{3}$

$\text{half } P_{x_1} + P_{x_1} = 1$

$\frac{3}{2} P_{x_1} = 1$

$\therefore P_{x_1} = \frac{2}{3}$

$r = \frac{1}{T_{avg}}$

$T_{avg} = P_{x_1} \cdot T_{x_1} + P_{x_2} T_{x_2} + \cancel{T_{gap}}$

$\cancel{\frac{2}{3} \cdot x_1 +}$

$= \frac{2}{3} \cdot 0.35 + \frac{1}{3} 0.7$

$\therefore r = 2.1428$

$= 0.466$

$H(S) = \sum_{k=1}^{n} P_k \log_2\left(\frac{1}{P_k}\right)$

$= P_{x_1} \log_2\left(\frac{1}{P_{x_1}}\right) + P_{x_2} \log_2\left(\frac{1}{P_{x_2}}\right)$

$= \frac{2}{3} \log_2\left(\frac{3}{2}\right) + \frac{1}{3} \log_2(3)$

$= 0.389975 + 0.52832$

$= 0.918295$ bits/Symbol.

Information rate $R = r\,H(S)$

$= 2.1428 \times 0.918295$

$= 1.967772$ bits/Symbol.

Extension of a discrete Memoryless Source.

Consider a primary source $S = \{S_1, S_2, S_3\}$ and the probability $P = \{P_1, P_2, P_3\}$, the binary source have 3 symbol and the no. of symbol is denoted as m ie; $m = 3$. With these symbols we are creating a new symbol source known as Extension source and that is equal to m^n.

If $n = 2 \Rightarrow 2^{nd}$ Order extension

n can be 2, 3 ... .

2^{nd} Order extension:

$m = 3 \quad n = 2$

extension $= 3^2 = 9$

$$\therefore \underset{\text{Source}}{S^2} = \begin{bmatrix} S_1S_1 & S_1S_2 & S_1S_3 \\ S_2S_1 & S_2S_2 & S_2S_3 \\ S_3S_1 & S_3S_2 & S_3S_3 \end{bmatrix}$$

The probability of the symbol in the extension source ie P(Symbols), extension source produces probability in the individual symbol.

$P_{S1} = P_1, \quad P_{S2} = P_2, \quad P_{S3} = P_3$

then, $P_{S_1S_2} = P_1 P_2$

Entropy is denoted as $H[S^n]$

ie; $\boxed{H[S^n] = n H[S]}$

Q A zero memory source has a source alphabet $S = S_1, S_2, S_3$ with the probability 0.5, 0.3, 0.2. Find the entropy of the source. Find the entropy of the second extension and verify the answer.

② $S^2 = 3^2 = 9$ $\begin{bmatrix} S_1S_1 & S_1S_2 & \dots \\ S_2S_1 & S_2S_2 & \dots \\ S_3S_1 & S_3S_2 & \dots \end{bmatrix}$

$P = [0.5\times0.5 \quad 0.5\times0.3 \dots]$

new $H[S^n] \Leftarrow$

$H[S^n] = \underbrace{n H[S]}$ } to verify

Sol: $S = \{S_1, S_2, S_3\}$

$P = \{0.5, 0.3, 0.2\}$

$$H[S] = \sum_{k=1}^{n} P_k \log_2\left(\frac{1}{P_k}\right)$$

$$= 0.5 \log_2 \frac{1}{0.5} + 0.3 \log_2 \frac{1}{0.3} + 0.2 \log_2 \frac{1}{0.5}$$

$$= 0.5 + 0.521 + 0.464$$

$$= 1.4853$$

Second order extension:

$$S^2 = 3^2 = 9 = \begin{bmatrix} S_1S_1 & S_1S_2 & S_1S_3 \\ S_2S_1 & S_2S_2 & S_2S_3 \\ S_3S_1 & S_3S_2 & S_3S_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.25 & 0.15 & 0.1 \\ 0.15 & 0.09 & 0.06 \\ 0.1 & 0.06 & 0.04 \end{bmatrix}$$

$$H[S^n] = 0.25 \log_2 \frac{1}{0.25} + 0.15 \log_2 \frac{1}{0.15} + 0.1 \log_2 \frac{1}{0.1}$$
$$0.15 \log_2 \frac{1}{0.15} + 0.09 \log_2 \frac{1}{0.09} + 0.06 \log_2 \frac{1}{0.06}$$
$$0.1 \log_2 \frac{1}{0.1} + \dots$$

$= 0.15 + 2(0.41054) + 2(0.332192) + 0.312651 + 2(0.24353) + 0.185754$

$= 2.48386$

$H[S^n] = n H[S]$

$= 2 \times 1.4853$

$= 2.9706$

2/24

Uniquelly Decodable and Prefix Code:

* The uniquelly decodable codes are the code having one to one relationship between the symbols.
* Every symbol in the symbol set has only one encoded code corresponding to bit.
* The necessary and sufficient condition for a code to be prefix, no complete codeword should come as a starting bit of anyother codeword.

Ex:

	Code 1	Code 2	Code 3
S_1	00	0	0
S_2	01	01	10
S_3	10	011	110
S_4	11	010	111

In the above example, code 2 is not a prefix code. The prefix code must satisfy the profit inequality.

All the above codes are uniquelly decodable, all the prefix codes are uniquelly decodable but uniquely decodable are not prefix.

imp.

Kraft's inequality.

Consider a discrete memoryless source with the source symbol $\{s_1, s_2, s_3 \ldots s_q\}$ with the probability of occurrence $P_1, P_2, P_3 \ldots P_q$ and the codeword length for the symbol are $l_1, l_2, \ldots l_q$, then the necessary and sufficient condition for the existance of instantaneous code with the code word length $l_1, l_2, \ldots l_q$ is that

$$\boxed{\sum_{k=1}^{q} r^{-l_k} \leq 1}$$

Where r is the no. of symbol in the code alphabet.

Ex: for binary value $r = 2$.

Proof:

Consider a ~~Square~~ Source with q symbols that is $S = s_1, s_2 \ldots s_q$ with the length $l_1, l_2 \ldots l_q$. Assume that there are r symbols in the code alphabet.

ie; $R = r_1, r_2 \ldots r_r$

For the instantaneous code to exist.

$$\sum_{k=1}^{q} r^{-l_k} \leq 1$$

Assume that codewords are instantaneous

$$n_1 < r$$

$$n_2 < (r - n_1) r$$

$$n_3 < [(r - n_1) r - n_2] r$$

$n_3 < (r^2 - n_1 r - n_2) r = n_3 < r^3 - n_1 r^2 - n_2 r$

$n_4 < r^4 - n_1 r^3 - n_2 r^2 - n_3 r$

If l is the maximum length of codeword,

$n_l \leq r^l - n_1 r^{l-1} - n_2 r^{l-2} \dots - n_{l-1} r$

multiplying each term by r^{-l},

$r^{-l} n_l \leq r^l - n_1 r^{l-1} \cdot r^{-l} - n_2 r^{l-2} r^{-l} \dots - n_{l-1} r \cdot r^{-l}$

$r^{-l} n_l \leq 1 - n_1 r^{-1} - n_2 r^{-2} \dots n_{l-1} r^{-l}$

$n_l r^{-l} + n_1 r^{-1} + n_2 r^{-2} \dots n_{l-1} r^{-l} \leq 1$

It can be written as,

$$\boxed{\sum_{j=1}^{l} n_j r^{-j} \leq 1}$$

n_j — positive integer.

Where n_j is the no. of codewords of length j.

Since n_l is the +ve integer the above equation can be written as

$$\sum_{j=1}^{l} n_j r^{-j} = \underbrace{[r^{-1} + r^{-1} + \dots r^{-1}]}_{n_1} + \underbrace{[r^{-2} + r^{-2} + \dots r^{-2}]}_{n_2} + \dots \underbrace{[r^{-l} + r^{-l} + \dots r^{-l}]}_{n_l}$$

$$\sum_{j=1}^{l} n_j r^{-j} = \sum_{j=1}^{n_1} r^{-1} + \sum_{j=1}^{n_2} r^{-2} + \sum_{j=1}^{n_3} r^{-3} + \dots \sum_{j=1}^{n_l} r^{-l}$$

$$\boxed{\sum_{k=1}^{q} r^{-l_k} \leq 1}$$

$$n_3 < (r^2 - n_1 r - n_2) r = n_3 < r^3 - n_1 r^2 - n_2 r$$

$$n_4 < r^4 - n_1 r^3 - n_2 r^2 - n_3 r$$

If l is the maximum length of codeword,

$$n_l \le r^l - n_1 r^{l-1} - n_2 r^{l-2} \dots - n_{l-1} r$$

multiplying each term by r^{-l},

$$r^{-l} n_l \le r^l - n_1 r^{l-1} \cdot r^{-l} - n_3 r^{l-2} r^{-l} \dots - n_{l-1} r \, r^{-l}$$

$$r^{-l} n_l \le 1 - n_1 r^{-1} - n_2 r^{-2} \dots n_{l-1} r^{-l}$$

$$n_l r^{-l} + n_1 r^{-1} + n_2 r^{-2} \dots n_{l-1} r^{-l} \le 1$$

It can be written as,

$$\boxed{\sum_{j=1}^{l} n_j r^{-j} \le 1}$$

n_j — positive integer.

Where n_j is the no. of condwords of length j.
Since n_l is the +ve integer the above equation can be written as

$$\sum_{j=1}^{l} n_j r^{-j} = \underbrace{[r^{-1} + r^{-1} + \dots r^{-1}]}_{n_1} + \underbrace{[r^{-2} + r^{-2} + \dots r^{-2}]}_{n_2} + \dots \underbrace{[r^{-l} + r^{-l} + \dots r^{-l}]}_{nl}$$

$$\sum_{j=1}^{l} n_j r^{-j} = \sum_{j=1}^{n_1} r^{-1} + \sum_{j=1}^{n_2} r^{-2} + \sum_{j=1}^{n_3} r^{-3} + \dots \sum_{j=1}^{n_l} r^{-l}$$

$$\boxed{\sum_{k=1}^{q} r^{-l_k} \le 1}$$

Q A Source emit one of the four Symbol S_0, S_1, S_2, S_3 with probability $\frac{1}{3}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}$. The Successive Symbol emitted by the Source are Statically independent. Calculate entropy of the Source.

Sol: $H[S] = \sum_{k=1}^{n} P_k \log_2 \frac{1}{P_k}$

$$H[S] = \frac{1}{3}\log_2 \frac{1}{1/3} + \frac{1}{6}\log_2 \frac{1}{1/6} + \frac{1}{4}\log_2 \frac{1}{1/4}$$

$$\frac{1}{4}\log_2 \frac{1}{1/4}$$

$$= 1.9591 \text{ bits/Symbol.}$$

Q Consider a Source with alphabet $S = \{x_1, x_2\}$ with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. Determine the entropy $H[S]$ of the Source. Write the Symbol of Second order extension of S and Determine the entropy $H[S^2]$. Verify $H[S^2] = 2H[S]$

Sol: $S = \{x_1, x_2\}$

$P = \{\frac{1}{4}, \frac{3}{4}\}$

$$H[S] = \sum_{k=1}^{n} P_k \log_2\left(\frac{1}{P_k}\right)$$

$$= \frac{1}{4}\log_2(4) + \frac{3}{4}\log_2\left(\frac{4}{3}\right)$$

$$= 0.5 + 0.31127$$

$$= 0.8112$$

Second order extension:

$$\# S = \begin{bmatrix} x_1 x_1 & x_1 x_2 \\ x_2 x_1 & x_2 x_2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.0625 & 0.1875 \\ 0.1875 & 0.5625 \end{bmatrix}$$

$$H[s^n] = 0.0625\log_2\frac{1}{0.0625} + 2\times 0.1875\log_2\frac{1}{0.1875} + 0.5625\log_2\frac{1}{0.5625}$$

$$= 0.25 + 0.90563 + 0.466917$$

$$= 1.62254$$

$$nH[s] = 2\times 0.8112$$
$$= 1.6224 \approx H[s^n]$$

Thus Verified.

2/24 The kraft inequality just tell us whether the instantaneous code exist or not. It does not tell us how to construct the code or does this gurantee that any code that has the Codeword length satisfying this inequality could be automatically instantaneous for binary code.

Q. Consider a discrete memoryless source with the symbol $x = 1, 2, 3, 4$. The following table shows 4 possible binary codes:

i) check whether the code satisfy kraft inequality

ii) Identify the instantaneous code.

iii) Show that code A and D are uniqually decodable but code B and C are not uniqualy decodable.

Symbol	Code A	Code B	Code C	Code D
x_1	00	0	0	0
x_2	01	10	11	100
x_3	10	11	100	110
x_4	11	110	110	111

Sol: $\sum_{k=1}^{q} r^{-l_k} \leq 1$

i) For Code A, $l_1 = l_2 = l_3 = l_4 = 2$

Since this is a binary code, $r = 2$ and $q = 4$

$\therefore \sum_{k=1}^{4} 2^{-l_k} \leq 1$

$2^{-l_1} + 2^{-l_2} + 2^{-l_3} + 2^{-l_4} \leq 1$

$= 2^{-2} + 2^{-2} + 2^{-2} + 2^{-2} \leq 1$

$= 1 \leq 1$

Satisfied.

Code B:

$l_1 = 1 \quad l_2 = 2 \quad l_3 = 2 \quad l_4 = 3$

$2^{-1} + 2^{-2} + 2^{-2} + 2^{-3} \leq 1$

$\frac{9}{8} > 1$

Code B doesnot Satisfied.

Code C:

$l_1 = 1 \quad l_2 = 2 \quad l_3 = 3 \quad l_4 = 3$

$2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} \leq 1$

$1 \leq 1$

Code C is Satisfied.

Code D:

$l_1 = 1 \quad l_2 = 3 \quad l_3 = 3 \quad l_4 = 3$

$$2^{-1} + 2^{-3} + 2^{-3} + 2^{-3} \leq 1$$

$$\frac{7}{8} \leq 1$$

Code D is Satisfied.

All the codes except code B, Satisfy kraft inequality.

ii) Code A and D are instantaneous code and uniqually decodable.

iii) Code B doesnot Satisfies kraft inequality
∴ It is not uniqually decodable.

Code C Satisfies kraft inequality but it is not uniqually decodable.
code A & D Satisfies kraft inequality and it is uniqually decodable.

Q Generate a binary prefix code for the Symbol with the probability 0.3, 0.2, 0.12, 0.25, 0.08, 0.05 also find the efficiency and reduntancy of the code.

Sol: Efficiency, $\boxed{\eta = \frac{H(S)}{\bar{L}}}$

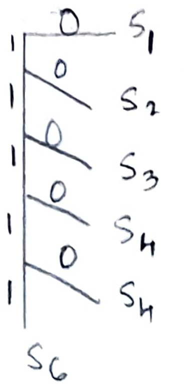
where $\bar{L} = \sum_{k} P_k l_k$

$\boxed{\gamma = 1 - \eta}$ Reduntancy.

Generation of binary prefix code.

Codeword:

Probability should be written in decreasing order.



$S_1 = 0$

$S_2 = 10$

$S_3 = 110$

$S_4 = 1110$

$S_5 = 11110$

$S_6 = 11111$

$S_1 : 0$

$S_2 = 10$

$S_3 = 110$

$S_4 = 1110$

$S_5 = 11110$

S_6 11111

Codeword length (l_k):

$S_1 = 1 \quad S_3 = 3 \quad S_5 = 5$

$S_2 = 2 \quad S_4 = 4 \quad S_6 = 5$

Symbol	Probability	Codeword	Codeword length
S_1	0.3	0	1
S_2	0.25	10	2
S_3	0.2	110	3
S_4	0.12	1110	4
S_5	0.08	11110	5
S_6	0.05	11111	5

decending order

$$\bar{L} = \sum_k P_k l_k$$

$$= 0.3 \times 1 + 0.25 \times 2 + 0.2 \times 3 + 0.12 \times 4 + 0.08 \times 5 + 0.05 \times 5$$

$$= 2.53$$

Efficiency, $\eta = \frac{H[S]}{\bar{L}}$

$$= \frac{2.3601}{2.53}$$

$\eta = 0.93$

Reduntancy, $r = 1 - \eta$

$r = 1 - 0.93$

$r = 0.0672$

|2|24 Procedure for prefix code problem:

1. Make a table with Symbol, probability codeword and codeword length
2. The probability should be in the descending order.
3. The codeword is formed by using code tree diagram. Draw the code tree diagram and find codeword for each symbol.
4. Write the codewordlength for each symbol ~~after~~
5. After completing the table, find $\bar{L}$ using the formula,

$$\bar{L} = \sum_{k} P_k l_k$$

6. Then, find $H[S]$ using the formula.

$$H[S] = \sum_{k=1}^{n} P_k \log_2 \left(\frac{1}{P_k}\right)$$

7. Find the efficiency using the equation,

$$\eta = \frac{H[S]}{\bar{L}}$$

8. Find the redundancy using the equation.

$$r = 1 - \eta$$

Consider a communication System $x_1, x_2, \dots x_n$ be the Symbols with the ranges transmitted by the Source (x). Let $y_1, y_2, \cdots y_n$ be the message received by the receiver y. So

$$x = \{x_1, x_2, \dots x_n\}$$

$$y = \{y_1, y_2, y_3 \cdots y_n\}$$

Let $P[x_1], P[x_2] \dots P[x_n]$ be the probability of transmitting Symbol $x_1, x_2, x_3, \dots x_n$ respectively. Let $P(y_1), P(y_2), \dots P(y_n)$ be the probability of receiving Signal $y_1, y_2, y_3 \cdots y_n$ respectively.

$$P(x_1) + P(x_2) + P(x_3) + \dots + P(x_n) = 1$$

Marginal Entropy:

$$H[X] = \sum_{k=1}^{m} P(x_k) \log_2 \frac{1}{P(x_k)}$$

$$H[Y] = \sum_{j=1}^{n} P(y_j) \log_2 \frac{1}{P(y_j)}$$

$H(X)$ and $H[Y]$ are the marginal entropy.

$$H[X,Y] = \sum_{k=1}^{m} \sum_{j=1}^{n} P(x_k\, y_j) \log_2 \left(\frac{1}{P(x_k\, y_j)}\right)$$

$$H[x|y] = \sum_{k=1}^{m} \sum_{j=1}^{n} P(x_k\, y_j) \log_2 \left(\frac{1}{P(x_k|y_j)}\right)$$

Q. If x and y are discrete random sources and $H[x, y]$ is their joint probability distribution and is given as $P(xy) = \{0.08, 0.05, 0.02, 0.05, 0.15, 0.07, 0.01, 0.12, 0.10, 0.06, 0.05, 0.04, 0.01, 0.12, 0.01, 0.06\}$. Calculate the marginal entropy, conditional entropy, and joint entropy and verify the properties of entropy.

Sol: $$P[x\,y] = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \\ \end{array} \begin{bmatrix} 0.08 + & 0.05 + & 0.02 + & 0.05 \\ 0.15 & 0.07 & 0.01 & 0.12 \\ 0.10 & 0.06 & 0.05 & 0.04 \\ 0.01 & 0.12 & 0.01 & 0.06 \end{bmatrix} = x_1$$
$$\qquad\qquad\quad y_1 \qquad y_2 \qquad y_3 \qquad y_4$$

$P[x]$ = Sum of rows $\quad P[y]$ = Sum of columns

$P[x_1] = 0.2 \quad P[x_2] = 0.35 \quad P[x_3] = 0.25 \quad P[x_4] = 0.2$

$\therefore P[x] = \{0.2, 0.35, 0.25, 0.2\}$

$P[y_1] = 0.34 \quad P[y_2] = 0.3 \quad P[y_3] = 0.09 \quad P[y_4] = 0.27$

$P[y] = \{0.34, 0.3, 0.09, 0.27\}$

$$H[x] = \sum_{k=1}^{m} P(x_k) \log_2 \frac{1}{P(x_k)}$$

$$= 0.2 \log_2 \frac{1}{0.2} + 0.35 \log_2 \frac{1}{0.35} + 0.25 \log_2 \frac{1}{0.25}$$
$$0.2 \log_2 \frac{1}{0.2}$$

$H[x] = 1.958$ bits/Symbol.

$$H[Y] = \sum_{j=1}^{n} P(y_j) \log_2 \frac{1}{P(y_j)}$$

$$= 0.34 \log_2 \frac{1}{0.34} + 0.3 \log_2 \frac{1}{0.3} + 0.09 \log_2 \frac{1}{0.09} + 0.27 \log_2 \frac{1}{0.27}$$

$H[Y] = 1.872$ bits/Symbol.

Joint Entropy

$$H[x, y] = \sum_{k=1}^{m} \sum_{j=1}^{m} P(x_k\, y_j) \log_2 \frac{1}{P(x_k\, y_j)}$$

$$= 0.08 \log_2 \frac{1}{0.08} + 3 \times 0.05 \log_2 \frac{1}{0.05} + 0.02 \log_2 \frac{1}{0.02} + 0.15 \log_2 \frac{1}{0.15} + 0.07 \log_2 \frac{1}{0.07} + 3 \times 0.01 \log_2 \frac{1}{0.01} + 2 \times 0.12 \log_2 \frac{1}{0.12} + 0.10 \log_2 \frac{1}{0.10} + 2 \times 0.06 \log_2 \frac{1}{0.06} + 0.04 \log_2 \frac{1}{0.04}$$

$$= 0.2915 + 0.6482 + 0.11287 + 0.41054 + 0.2685 + 0.1993 + 0.73413 + 0.33219 + 0.48706 + 0.18575$$

$H[x, y] = 3.67004$ bits/Symbol.

Conditional Entropy

$$H[x|y] = \sum_{k=1}^{m} \sum_{j=1}^{n} P(x_k\, y_j) \log_2 \left(\frac{1}{P(x_k | y_j)} \right)$$

$$P\left[\frac{x_k}{y_j}\right] = \frac{P[x_k\, y_j]}{P[y_j]}$$

$$H[Y/X] = \sum_{k=1}^{m} \sum_{j=1}^{n} P(x_k\, y_j) \log_2 \frac{1}{P(y_j/x_k)}$$

$$P[Y/X] = \frac{P[x_k\, y_j]}{P(x)}$$

$$P[Y/X] = \begin{bmatrix} 0.4 & 0.25 & 0.1 & 0.25 \\ 0.4285 & 0.2 & 0.0285 & 0.3428 \\ 0.4 & 0.24 & 0.2 & 0.16 \\ 0.05 & 0.6 & 0.05 & 0.3 \end{bmatrix}$$

$$H[Y/X] = 0.08 \log_2 \frac{1}{0.4} + 0.05 \log_2 \frac{1}{0.25} + 0.02 \log_2 0.$$
$$0.05 \log_2 \frac{1}{0.25} + 0.15 \log_2 \frac{1}{0.4285} + 0.07 \log_2 \frac{1}{0.2} +$$
$$0.01 \log_2 \frac{1}{0.0285} + 0.12 \log_2 \frac{1}{0.3428} + 0.10 \log_2 \frac{1}{0.4} +$$
$$0.06 \log_2 \frac{1}{0.24} + 0.05 \log_2 \frac{1}{0.2} + 0.04 \log_2 \frac{1}{0.16} +$$
$$0.01 \log_2 \frac{1}{0.05} + 0.12 \log_2 \frac{1}{0.6} + 0.01 \log_2 \frac{1}{0.05} +$$
$$0.06 \log_2 \frac{1}{0.3}$$

$$= 0.1057 + 0.1 + 0.06643 + 0.1 + 0.18339 + 0.1625 +$$
$$0.05132 + 0.1853 + 0.13219 + 0.1235 + 0.11609$$
$$0.10575 + 0.04321 + 0.08843 + 0.04321 + 0.104$$

$H[Y/X] = \underline{1.7108} \simeq 1.7108$ bits/Symbol

Properties of Entropy:

$$H[X\,Y] = H[X] + H[Y/X]$$

$$H[X\,Y] = H[Y] + H[X/Y]$$

$$H[X] \geq H[X/Y]$$

$$H[Y] \geq H[Y/X]$$

$$H[X, Y] \leq H[X] + H[Y]$$

* $H[x] + H[y/x] = 1.958 + 1.7108$

$= 3.6688 \approx 3.67004$ bits/symbol

Thus, $H[x, y] = H[x] + H[y/x]$

* $H[y] + H[x/y] = 1.872 + 1.81$

$= 3.682 \approx 3.67004$ bits/Symbol.

Thus, $H[x\,y] = H[y] + H[x/y]$

* $H[x] = 1.958 \geq H[x/y] = 1.81$
* $H[y] = 1.872 \geq H[y/x] = \cancel{1.6966}\ 1.7108$
* $H[x, y] = 3.67004 \leq H[x] + H[y] = 3.83.$

2/24

Q. Verify the properties of entropy for a discrete memoryless channel defined by the channel matrix or transition matrix.

$$P[Y/x] = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}$$

Given $P[x_1] = 0.25$, $P[x_2] = 0.35$, $P[x_3] = 0.4$

Sol: M.E

$$H[x] = \sum_{k=0}^{m} P(x_k) \log_2 \left(\frac{1}{P(x_k)}\right)$$

$$= 0.25 \log_2 \frac{1}{0.25} + 0.35 \log_2 \frac{1}{0.35} + 0.4 \log_2 \frac{1}{0.4}$$

$$= 0.5 + 0.5301 + 0.5287$$

$$= 1.5588 \text{ bits/Symbol}$$

$\therefore H[x, y] = P[\frac{y}{x}] . P[x]$

$$P[x, y] = \begin{bmatrix} 0.05 & 0.075 & 0.125 \\ 0.105 & 0.14 & 0.105 \\ 0.16 & 0.08 & 0.16 \end{bmatrix}$$

$$P[y] = \{0.315 \quad 0.295 \quad 0.39\}$$

$$H[y] = \sum_{k=1}^{m} P(y_j) \log_2 \frac{1}{P(y_j)}$$

$$= 0.315 \log_2 \frac{1}{0.315} + 0.295 \log_2 \frac{1}{0.295} + 0.39 \log_2 \frac{1}{0.39}$$

$$= 0.52497 + 0.51955 + 0.52979$$

$$= \underline{\underline{1.5743}} \text{ bits/Symbol.}$$

$$H[x, y] = \sum_{k=1}^{m} \sum_{j=1}^{n} P(x_k, y_j) \log_2 \frac{1}{P(x_k, y_j)}$$

$$= 0.05 \log_2 \frac{1}{0.05} + 0.075 \log_2 \frac{1}{0.075} + 0.125 \log_2 \frac{1}{0.125} + 2 \times 0.105 \log_2 \frac{1}{0.105} + 0.14 \log_2 \frac{1}{0.14} + 2 \times 0.16 \log_2 \frac{1}{0.16} + 0.08 \log_2 \frac{1}{0.08}$$

$$= 0.2160 + 0.28027 + 0.375 + 0.6828 + 0.3971 + 0.84603 + 0.2915$$

$$H[x\ y] = \underline{\underline{3.0833}} \text{ bits/Symbol.}$$

$$H[y/x] = \sum_{k=1}^{m} \sum_{j=1}^{m} P(x_k\ y_j) \log_2 \frac{1}{P(y_j/x_k)}$$

$$= 0.05 \log_2 \frac{1}{0.2} + 0.075 \log_2 \frac{1}{0.3} + 0.125 \log_2 \frac{1}{0.5} + 0.105 \log_2 \frac{1}{0.3} + 0.14 \log_2 \frac{1}{0.4} + 0.105 \log_2 \frac{1}{0.3} + 0.16 \log_2 \frac{1}{0.4} + 0.08 \log_2 \frac{1}{0.2} + 0.16 \log_2 \frac{1}{0.4}$$

$$= 0.1160 + 0.1302 + 0.125 + 0.36476 + 0.18506 + 0.4230 + 0.18575$$

$$= \cancel{1.3995}\ \underline{1.40477} \text{ bits/Symbol}$$

$$P[x/y] = \frac{P[x,y]}{P[y]} = \begin{bmatrix} 0.1587 & 0.254 & 0.320 \\ 0.333 & 0.474 & 0.269 \\ 0.507 & 0.271 & 0.410 \end{bmatrix}$$

$$H[x/y] = \sum_{k=1}^{m} \sum_{j=1}^{n} P[x_k, y_j] \log_2 \frac{1}{P[x_k/y_j]}$$

$$= 0.05 \log_2 \frac{1}{0.1587} + 0.075 \log_2 \frac{1}{0.254} + 0.125 \log_2 \frac{1}{0.320}$$

$$+ 0.105 \log_2 \frac{1}{0.333} + 0.14 \log_2 \frac{1}{0.474} + 0.105 \log_2 \frac{1}{0.269}$$

$$+ 0.16 \log_2 \frac{1}{0.1507} + 0.08 \log_2 \frac{1}{0.271} + 0.16 \log_2 \frac{1}{0.410}$$

$$= 0.1327 + 0.1482 + 0.2054 + 0.1664 +$$

$$0.1507 + \cancel{0.2841} + 0.1789 + 0.1567 + 0.1506 +$$

$$0.2058$$

$= 1.5154$ bits/Symbol.

$* \; H[x,y] = 3.0833 = H[y] + H[x/y] = 3.089$ bits/Symbol

$* \; H[x,y] = H[x] + H[y/x] = 3.083$ bits/Symbol.

$* \; H[x] = 1.5588 \geq H[x/y] = 1.5154$ bits/Symbol.

$* \; H[y] = 1.5743 \geq H[y/x] = 1.4047$ bits/Symbol.

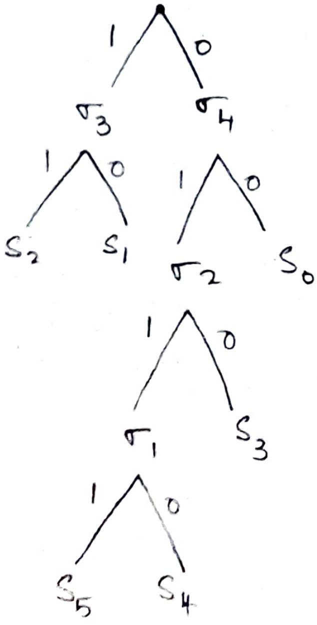
$* \; H[x,y] = 3.0833 \leq H[x] + H[y] = 3.1331$ bits/Symbol

7/2/24 Huffman coding.

Q To generate the Huffman code for the symbols with the probability 0.3, 0.2, 0.12, 0.25, 0.08, 0.05. Find the efficiency and redundancy of the code.

Sol:

Symbol	Probability				
S_0	0.3	0.3	0.3	$0.45 = \sigma_3$	$\sigma_4 = 0.55$
S_1	0.25	0.25	$\sigma_2 = 0.25$	S_0 0.3	$\sigma_3 = 0.45$
S_2	0.2	0.2	S_1 0.25	σ_2 $0.25 = \sigma_2$	
S_3	0.12	σ_1 0.13	S_2 0.2	$\sigma_4 = 0.55$	
S_4	S_4 0.08	S_3 0.12	$\sigma_3 = 0.45$		
S_5	S_5 0.05 σ_1 $= 0.13$	$\sigma_2 = 0.25$			



Codeword length

$S_0 = 00 = 2$

$S_1 = 10 = 2$

$S_2 = 11 = 2$

$S_3 = 010 = 3$

$S_4 = 0010 = 4$

$S_5 = 0111 = 4$

$$\bar{L} = \sum_k P_k l_k$$

$$= 0.3 \times 2 + 0.25 \times 2 + 0.2 \times 2 + 0.12 \times 3 + 0.084 \times 4 + 0.05 \times 4$$

$$\bar{L} = 2.38$$

$$H[S] = \sum_{k=1}^{n} P_k \log_2 \frac{1}{P_k}$$

$$= 0.3 \log_2 \frac{1}{0.3} + 0.25 \log_2 \frac{1}{0.25} + 0.2 \log_2 \frac{1}{0.2} +$$
$$0.12 \log_2 \frac{1}{0.12} + 0.08 \log_2 \frac{1}{0.08} + 0.05 \log_2 \frac{1}{0.05}$$

$$= 0.5210 + 0.5 + 0.4643 + 0.36706 + 0.2915 +$$
$$0.21609$$

$$H[S] = 2.359 \text{ bits/Symbol.}$$

Efficiency, $\eta = \frac{H(S)}{\bar{L}} = \frac{2.359}{2.38}$

$$= 0.991$$
$$= 99.1\%$$

Redundancy $= 1 - \eta$
$= 0.0099$
$= 9 \times 10^{-3}$

Uncertainity Surprise and Information.

Output of a random experiment is described by Source $S = \{S_0, S_1, \dots S_K\}$ S is the random Variable. $P(S = S_K) = P_K$.
The event $S = S_K$ describes the emission of Symbols S_K with probability P_K. Before that event occurs there will be an amount of uncertainity. After that event occurs there will be gain in Information. When the event occurs, there will be amount of Surprise.

Self Information.

Communication System involves the messages $S = \{S_0, S_1 \dots S_K\}$ with probabi[lity] $P = P_0, P_1 \dots P_K$. Let the transmitter to send the message S_K with probability P_K which contains the information I_K.

* Self information is the logarithm function of a reciprocal of probability P_K of the event

$$S = S_K, \quad I(S_K) = \log_2\left(\frac{1}{P_K}\right)$$

$$= \log_2 (P_K)^{-1} \text{ bits}$$

$$= -\log_2 (P_K) \text{ bits}$$

If the base is 2, unit is bits.

If the base is 10, unit is Hartley or Decit[ly]

If $\ln \rightarrow$ unit = nat

Properties of Self information:

1. $I(S_K) = 0$ for $P_K = 1$

It indicates if the outcome of an event is certain even before it occurs, then there is no information bits.

2. $I(S_K) \geq 0$ for $0 \leq P_K \leq 1$

3. $I(S_K) > I(S_i)$ for $P_K < P_i$

4. $I(S_K S_i) = I(S_K) + I(S_i)$

5. $I(S_K) = \infty$ for $P_K = 0$

$I(S_K) = \log_2 \frac{1}{P_i}$

Graph - Self information

imp ✓

Source Coding

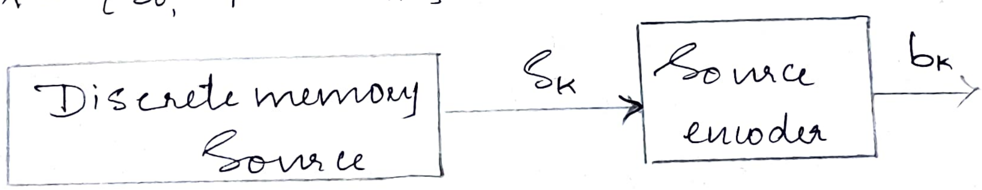
The efficiency of operation of a communication system is improved by transmitting a coded signal.

Efficient coding techniques are used to detect errors and to correct errors caused by noise.

Source Coding Theorem:

It is used to represent the data generated by discrete source efficiently. The device that performs this implementation is called source encoders. It needs the statistics of the source there by assigning short code words to frequent source symbols and long code words to rare source symbols.

$X = \{S_0, S_1 \dots S_K\}$



Functional requirements of efficient source encoder.

* The code words produced by encoder are in binary form.
* The source code is uniquely decodable as uniquely decipherable, so that the original source sequence can be reconstructed perfectly.

Discrete Memoryless Source emits equally likely binary Symbols in every t Seconds. The output of the DMS (S_K) is converted by the Source encoder into a block of zeros and ones denoted by b_K.

The Symbol S_K is generated with probability P_K. The binary code word length is the assigned to Symbols. Then the code word length to the Source encoder is $\boxed{L = \sum_{k=0}^{k-1} P_K l_K}$

Code Efficiency:

It is defined as the ratio of L_{min} to the average code word length L.

$$\boxed{\eta = \frac{L_{min}}{L}}$$

$L_{min} \leq L$

$\eta \leq 1$

Huffman Coding

This coding assigns a Sequence of bits to each Symbol of an alphabet. It is one of the prefix code.

Coding Algorithm:

1. List the Source Symbols in the order of decreasing probabilities.
2. Splitting: It assigns zero and one to the two Source Symbols of lowest probability.
3. Combine these two Source Symbols to a

new Source Symbol with probability equal to the Sum of two original probabilities.

4. Place the new Symbol in the list in accordance with its probability Value.
5. Repeat the procedure until the final list contains only two Symbols.
6. Assign '0' and '1' to these Symbols.
7. Read the codeword of each Symbol from the last Splitting Stage (ss).

Properties of Entropy with Proof.

Entropy : Average information content of Symbols.

Consider that there are $M = \{m_1, m_2, m_3 \cdots\}$ different messages with probabilities $P = \{P_1, P_2, P_3 \cdots\}$. Suppose that a Sequence of 'L' messages is transmitted.

Let the message be in the form of

$(m_4, m_2), (m_1, m_3), \cdots (m_1, m_M) \cdots$

respective probabilities;

$P_1, P_1, \cdots P_1$

$P_1 L$ messages of m_1 are transmitted

$P_2 L$ messages of m_2 are transmitted.

- - - -

- - - -

$P_M L$ messages of m_M are transmitted

We know that $I(m_1) = \log_2\left(\frac{1}{P_1}\right)$

If $(P_1 L)$ messages of m_1 are transmitted

$I_1(\text{total}) = P_1 L \log_2\left(\frac{1}{P_1}\right)$

$I(\text{total}) = P_1 L \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + \dots P_M L \log_2$

$$\text{Average of Information} = \frac{\text{Total information}}{\text{No. of information}}$$

$$= \frac{P_1 L \log_2(1/P_1) + P_2 L \log_2(1/P_2) + \dots + P_M L \log_2(1/P_M)}{L}$$

$$= \frac{\cancel{L}\left[P_1 \log_2(1/P_1) + P_2 \log_2(1/P_2) + \dots + P_M \log_2(1/P_M)\right]}{\cancel{L}}$$

$$= P_1 \log_2(1/P_1) + P_2 \log_2(1/P_2) + \dots + P_M \log_2(1/P_M)$$

$$\boxed{\text{Entropy, } H = \sum_{k=1}^{M} P_k \log_2(1/P_k)}$$

imp **Properties:**

1. Entropy is zero if the event is sure

$$H = 0 \to P_k = 1, \quad P_k = 0$$

2. When $P_k = 1/M$ for all 'M' symbols, these symbols are equally likely.

$$H = \log_2 M$$

3. Upper bound on entropy is given as

$$H_{max} \le \log_2 M$$

Proof :

1. Entropy is zero if the event is sure.

If $P_K = 1$; then

Entropy $H[S] = \sum_{k=1}^{M} P_K \log_2 (1/P_K)$

$$= \sum_{k=1}^{M} \log_2 (1/1)$$

$$H[S] = 0$$

If $P_K = 0$; then

Entropy $H = \sum_{k=1}^{M} P_K \log_2 (1/P_K)$

$$= \sum_{k=1}^{M} 0 \times \log_2 (1/P_K)$$

$$H = 0$$

Hence proved.

2. When $P_K = 1/M$ for all 'M' symbols, then symbols are equally likely.

$$H = \sum_{k=1}^{M} P_K \log_2 (1/P_K)$$

$$= \sum_{k=1}^{M} \frac{1}{M} \log_2 (1/1/M)$$

$$= \sum_{k=1}^{M} \frac{1}{M} \log_2 (M) = \frac{1}{M} [\log_2 M + \log_2 M + \dots + \log_2 M]$$

$$= \frac{1}{M} \log_2 M^M$$

$$= \frac{M}{M} \log_2 M$$

$$\boxed{H = \log_2 M}$$

3. Upper bound on entropy is given as $H_{max} = \log$

To prove this consider basic property of natural logarithm is $\ln x \leq x - 1$ — ①

$\ln = \log_e()$ and consider two probability distributions $P_1, P_2 \dots P_k$, $q_1, q_2 \dots q_k$ on the alphabet $X = \{x_1, x_2, x_3 \dots x_k\}$

Consider $$\sum_{k=1}^{M} P_k \log\left(\frac{q_k}{P_k}\right) = \sum_{k=1}^{M} \frac{P_k \log_{10}\left(\frac{q_k}{P_k}\right)}{\log_{10} 2} \cdot \frac{\log_{10}}{\log_{10}}$$

$$= \sum_{k=1}^{M} P_k \frac{\log_{10}\left(\frac{q_k}{P_k}\right)}{\log_{10} e} \cdot \frac{\log_{10} e}{\log_{10} 2} = \sum_{k=1}^{M} P_k \log_2 e \log_e\left(\frac{q_k}{P_k}\right)$$

$$= \log_2 e \sum_{k=1}^{M} P_k \ln \frac{q_k}{P_k} \quad \text{—— ②}$$

If $x = \frac{q_k}{P_k}$ then from ①

$$\ln \frac{q_k}{P_k} \leq \frac{q_k}{P_k} - 1$$

The equation ② can be written as,

$$\sum_{k=1}^{M} P_k \log_2 \frac{q_k}{P_k} \leq \log_2 e \sum_{k=1}^{M} P_k \left(\frac{q_k}{P_k} - 1\right)$$

$$\leq \log_2 e \sum_{k=1}^{M} \cancel{P_k} \left(\frac{q_k - P_k}{\cancel{P_k}}\right)$$

$$\leq \log_2 e \left[\sum_{k=1}^{M} q_k - \sum_{k=1}^{M} P_k\right]$$

Here $\sum_{k=1}^{M} q_k = \sum_{k=1}^{M} P_k = 1$ [Sum of probabilities = 1]

Therefore, $$\sum_{k=1}^{M} P_k \log_2 \frac{q_k}{P_k} \leq \log_2 e \,(0)$$

$$= \sum_{k=1}^{M} P_k \log_2 \frac{q_k}{P_k} \leq 0$$

$$\sum_{k=1}^{M} P_k \log_2 q_k - \sum_{k=1}^{M} P_k \log_2 P_k \leq 0$$

$$\sum_{k=1}^{M} P_k \log_2 q_k \leq \sum_{k=1}^{M} P_k \log_2 P_k$$

$$H \leq \left[P_1 \log_2 1/q_1 + P_2 \log_2 1/q_2 + \dots + P_M \log_2 1/q_M\right]$$

take $P_1, P_2, P_3 \dots = 1/M$

$q_1, q_2, q_3 \dots = 1/M$

$$H \leq \left[\frac{1}{M} \log_2 \frac{1}{1/M} + \frac{1}{M} \log_2 \frac{1}{1/M} + \dots + \frac{1}{M} \log_2 \frac{1}{1/M}\right]$$

$$H \leq \frac{1}{M}\left[\log_2 M + \log_2 M + \dots + \log_2 M\right]$$

$$H \leq \frac{1}{M} \log_2 M^M$$

$$H \leq \frac{M}{M} \log_2 M$$

$$\boxed{H \leq \log_2 M}$$ Hence Proved.

→ Prove $H[x, y] = H[x] + H[y/x]$

$$H[x, y] = \sum_j \sum_k P(x_j y_k) \log_2 \left(\frac{1}{P(x_j y_k)}\right)$$

$$P(x_j y_k) = P(y_k/x_j)\, P(x_j)$$ $\log ab = \log a + \log b$

$$P(x_j y_k) = P(x_j/y_k)\, P(y_k)$$

$$H(x, y) = \sum_j \sum_k P(x_j, y_k) \log_2 \frac{1}{P(y_k/x_j)\, P(x_j)}$$

$$= \sum_j \sum_k P(x_j y_k) \left[\log_2 \frac{1}{P(y_k/x_j)} + \log_2 \frac{1}{P(x_j)}\right]$$

$$= \sum_j \sum_k P(x_j y_k) \log_2 \frac{1}{P(y_k/x_j)} + \sum_j \sum_k (P(x_j, y_k)) \log_2 \frac{1}{P(x_j)}$$

$$= H[y|x] + \sum_j \sum_k P(y_k|x_j)\, P(x_j) \log_2 \frac{1}{P(x_j)}$$

$$= H[y|x] + \sum_j P(x_j) \log_2 \frac{1}{P(x_j)}$$

$$H(x,y) = H[y|x] + H(x)$$

$\Rightarrow$ Prove $H[x,y] = H[y] + H[x|y]$

$$H[x,y] = \sum_j \sum_k P(x_j\, y_k) \log_2 \frac{1}{P(x_j, y_k)}$$

$$P(x_j, y_k) = P(x_j|y_k)\, P(y_k)$$

$$H(x,y) = \sum_j \sum_k P(x_j, y_k) \log_2 \frac{1}{P(x_j|y_k)\, P(y_k)}$$

$$= \sum_j \sum_k P(x_j\, y_k) \log_2 \frac{1}{P(x_j|y_k)} + \sum_j \sum_k P(x_j\, y_k) \log_2 \frac{1}{P(y_k)}$$

$$= H(x|y) + \sum_j \sum_k P(x_j|y_k)\, P(y_k) \log_2 \frac{1}{P(y_k)}$$

$$= H(x|y) + \sum_j P(x_j|y_k) \sum_k P(y_k) \log_2 \frac{1}{P(y_k)}$$

$$H(x,y) = H[x|y] + H[Y]$$

Joint Entropy:

Entropy of joint probability distribution

$P(x,y) =$ Joint probability distribution.

$$\boxed{H[x,y] = \sum_{i=1}^{m} \sum_{j=1}^{m} P(x_i\, y_j) \log_2 \left[\frac{1}{P(x_i, y_j)}\right]}$$

Conditional Entropy:

Average conditional self information is called as conditional entropy.

$$\boxed{\begin{aligned} H[x|y] &= \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\, y_j)\log_2\left[\frac{1}{P(x_i|y_j)}\right] \\ H[y/x] &= \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j)\log_2\left[\frac{1}{P(y_j/x_i)}\right] \end{aligned}}$$

Relationship:

$$H(x,y) = H[x|y] + H[y]$$
$$= H[y/x] + H[x]$$

Mutual Information:

It is defined as the amount of information transferred where x_i is transmitted and y_j is received.

$$x_i \longrightarrow \boxed{\text{Comm}^n} \longrightarrow y_j$$

$$MI = I(x_i\, y_j) = \log_2\left[\frac{P(x_i|y_j)}{P(x_i)}\right] \quad \text{--- (1)}$$

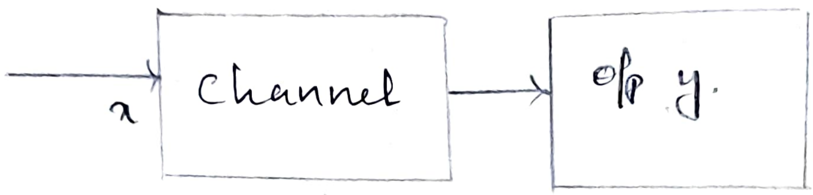
Properties:

1. Mutual Information is Symmetric
$$I(x,y) = I(y,x)$$
2. Mutual information is always +Ve ; $I(x,y) \geq 0$
3. Mutual information may be expressed as entropies
$$MI = H[x] - H[x|y]$$
$$= H[y] - H[y/x]$$
4. Mutual information is related to joint entropy.
$$I[x,y] = H[x] + H[y] - \underbrace{H[x,y]}_{\text{joint entropy}}$$

2/02/24

Module - 2

Discrete Memoryless Channel



Discrete Memoryless Channel:

A discrete memoryless channel is a statistical model with i/p x and o/p y that is a noisy version of x, both x & y are random variable.

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{j-1} \end{bmatrix} \Rightarrow \text{Input Signal} \xrightarrow[P[y_k/x_j]]{\text{channel}} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{k-1} \end{bmatrix} = y \quad (\text{o/p Signal})$$

The channel is discrete & memoryless, the transition probability matrix / channel probability matrix / conditional probability matrix is given by:

$$P[y/x] = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) & \cdots & P(y_n/x_1) \\ P(y_1/x_2) & P(y_2/x_2) & \cdots & P(y_n/x_2) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P(y_1/x_m) & P(y_2/x_m) & \cdots & P(y_n/x_m) \end{bmatrix}$$

ie; $\sum_{j=1} P(y_j/x_k) = 1$, for all k

Mutual Information:

$$\boxed{\begin{aligned} I[x, y] &= H[x] - H[x|y] \\ I[x, y] &= H[y] - H[y|x] \end{aligned}}$$

The channel capacity is denoted as C and

$$\boxed{C = Max(I[x, y])}$$

Prove ~~that~~ the properties of $I[x, y]$

1. $I[x, y]$ is non-negative ie;

$$I[x, y] \geq 0$$

2. $I[x, y]$ is Symmetric

$$I[x, y] = I[y, x]$$

The efficiency of the channel is calculated as

$$\eta = \frac{I[x, y]}{C} \times 100$$

1. $I[x,y] \geq 0$

$$I[x,y] = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} \quad \text{—— ①}$$

$$P(x_i/y_j) = \frac{P(x_i, y_j)}{P(y_j)}$$

Substitute in ①;

$$I[x,y] = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left(\frac{P(x_i, y_j)}{P(x_i) P(y_j)} \right)$$

$$I[x,y] = -\sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left[\frac{P(x_i) P(y_j)}{P(x_i, y_j)} \right]$$

$$-I[x,y] = \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{P(x_i, y_j)}_{P_K} \log_2 \left[\frac{P(x_i) P(y_j)}{P(x_i, y_j)} \right]^{q_K}$$

$$\sum_{k=1}^{m} P_K \log_2 \left(\frac{q_K}{P_K} \right) \leq 0$$

$$-I[x,y] \leq 0$$

$$I[x,y] \geq 0$$

Hence proved.

2. Mutual Information Symmetric property.

$$I[x,y] = I[y,x]$$

$$P(x_i\ y_j) = P(x_i/y_j)\, P(y_j) \quad \text{—— ①}$$

$$P(x_i\ y_j) = P(y_j/x_i)\, P(x_i) \quad \text{—— ②}$$

① = ②

$$P(x_i/y_j)\, P(y_j) = P(y_j/x_i)\, P(x_i)$$

$$\frac{P(x_i/y_j)}{P(x_i)} = \frac{P(y_j/x_i)}{P(y_j)} \quad \text{—— ③}$$

$$I[x,y] = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i\ y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}$$

$$I[y,x] = \sum_{j=1}^{m} \sum_{i=1}^{n} P(y_j, x_i) \log_2 \frac{P(y_j/x_i)}{P(y_j)}$$

$$I(x, y) = \sum_{i=1}^{n} \sum_{j=m}^{n} P(x_i, y_j) \log_2 \frac{P(y_j/x_i)}{P(y_j)}$$

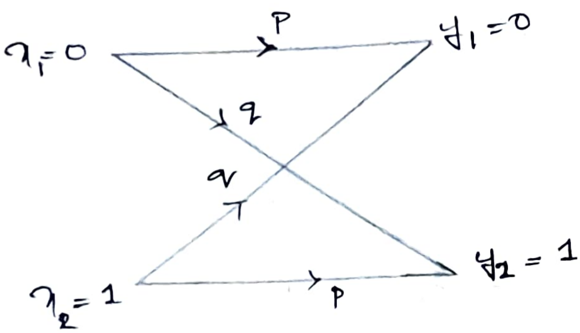
$$\therefore I[x, y] = I[y, x]$$

Binary ~~Symbol~~ Symmetric channel [BSE] and
Binary ~~Irrational~~ Erasure channel [BRE]

BSC:
equal no. of i/p o/p

BSC:

P is the probab. of error and
q is the probab. of non-error.



$$P[y|x] = \begin{array}{cc} 00 & 01 \\ \end{array} \begin{bmatrix} p & q \\ q & p \end{bmatrix}$$

$p + q = 1$

$q = 1 - p$

Let us assume that the Source probabilities are given by, $P(x_1) = \alpha$ and $P[x_2] = 1 - \alpha$

$$\therefore P[x, y] = P[y|x] . P[x]$$

$$P[x, y] = \begin{bmatrix} p\alpha & q\alpha \\ q(1-\alpha) & p(1-\alpha) \end{bmatrix}$$

$$H[y|x] = \sum_{k=1}^{n} \sum_{j=1}^{m} P[x_k, y_j] \log_2 \frac{1}{P(y_j|x_k)}$$

$$= p\alpha \log_2 \frac{1}{p} + q\alpha \log_2 \frac{1}{q} + \cancel{q(1-\alpha)}$$
$$q(1-\alpha) \log_2 \frac{1}{q} + p(1-\alpha) \log_2 \frac{1}{p}$$

$$= p\alpha \log_2 \frac{1}{p} + q\alpha \log_2 \frac{1}{q} +$$
$$q \log_2 \frac{1}{q} - q\alpha \log_2 \frac{1}{q} + p \log_2 \frac{1}{p}$$

$-P \times \log_2 \frac{1}{P}$

$$H[Y/X] = P\log_2 \frac{1}{P} + q\log_2 \frac{1}{q}$$

$$I[X,Y] = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i\, y_j)\log_2 \frac{P(x_i/y_j)}{P(x_i)}$$

$$I[X,Y] = H[Y] - H[Y/X]$$

Capacity of channel $C = \max(I(X,Y))$

$H[Y] = \log_2 n$ $\quad n = 2$ $\quad$ n = no. of messages. $H[Y]$ = max. Entropy

$\therefore H[Y] = \log_2 2$

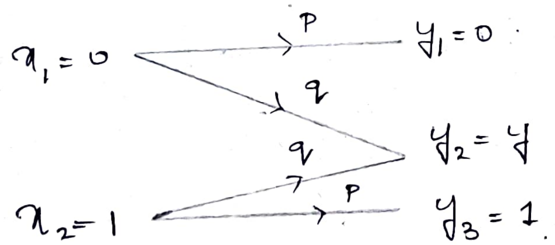
$$\therefore I[X,Y] = \log_2 2 - \underbrace{\left[P\log_2 \frac{1}{P} + q\log_2 \frac{1}{q}\right]}_{\text{Length of entropy} = h}$$

$$I(X,Y) = \log_2 2 - h$$

$\therefore$ Capacity $= \max[\log_2 2 - h]$ $\quad \log_2 2 = 1.$

2/24 $\boxed{C = 1 - h.}$ capacity of binary Symmetric channel.

Binary Erasure channel



$$P[Y/X] = \begin{bmatrix} P & q & 0 \\ 0 & q & P \end{bmatrix}$$

Let us assume that the source probabilities are given by, $P(x_1) = \alpha$ and $P(x_2) = 1 - \alpha$

$$P[X,Y] = P[Y/X] \cdot P[X]$$

$$= P\alpha \log_2 \frac{1}{1} + q\alpha \log_2 \frac{1}{\alpha} + q \log_2 \frac{1}{1-\alpha} - \alpha \log_2 \frac{1}{1-\alpha}$$
$$+ P \log_2 \frac{1}{1} - \alpha \log_2 \frac{1}{1}$$

$$= \alpha q \log_2 \frac{1}{\alpha} + q(1-\alpha) \log_2 \frac{1}{1-\alpha}$$

$$= q \left[\underbrace{\alpha \log_2 \frac{1}{\alpha} + (1-\alpha) \log_2 \frac{1}{1-\alpha}}_{\text{Entropy of i/p}} \right]$$

$$= q H(x)$$

$$H(x|y) = q H(x)$$

$$\therefore I(x,y) = H[x] - H[x|y]$$
$$H[x] - qH[x]$$

$P + q = 1$

$\underline{P = 1 - q}$

$$I(x,y) = H[x][1-q]$$

Capacity $C = \max[I(x,y)]$

$$= \max[H(x)(\underbrace{1-q}_{\text{quantity of error}})]$$

$$C = \max[H(x) \cdot P]$$

$$C = \max[H(x) \cdot P]$$

$m = 2$

$$= P \log_2 m$$

$$\boxed{C = P}$$

Q A binary channel has the following noise characteristics $P[y/x] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$. If the i/p symbols are transmitted with the probability $3/4$ and $1/4$ respectively.

a) find $I[x,y]$

b) Compute channel capacity

c) What are the i/p symbol probability that correspon to channel capacity.

Sol:

a) $P[y|x] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$ $\quad P[x] = \{3/4, 1/4\}$

$P[x,y] = P[y|x] \cdot P[x]$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix}$

$H[y|x] = \sum_j \sum_k P(x,y) \log_2 \frac{1}{P(y|x)}$

$= \overbrace{\frac{1}{2} \log_2 \frac{1}{2/3}}^{0.2924} + \overbrace{\frac{1}{4} \log_2 \frac{1}{1/3}}^{0.3962} +$

$\overbrace{\frac{1}{12} \log_2 \frac{1}{1/3}}^{0.13208} + \overbrace{\frac{1}{6} \log_2 \frac{1}{2/3}}^{0.38997}$

$= 0.92$ bits/Symbol.

$P(y) = \left[\frac{7}{12}, \frac{5}{12}\right]$

$\therefore H[y] = \sum_k P(y) \log_2 \frac{1}{P(y)}$

$= \frac{7}{12} \log_2 \frac{12}{7} + \frac{5}{12} \log_2 \frac{12}{5}$

$= 0.4536 + 0.52626$

$= 0.9798$ bits/Symbol.

$I[x,y] = H[y] - H[y|x]$

$= 0.9798 - \cancel{0.9798}\ 0.92$

$= 0.059$

b) $C = \max(I(x,y))$

$= \cancel{H(y)_{max}} \; \log_2 M - H[y/x]$ $\quad M = 2$ (i/ps)

$= \log_2 2 - 0.92$

$= 1 - 0.92$

$C = \underline{\underline{0.08}}$

c) Inorder to transmit the information equal to the channel capacity, the entropy should be maximum $[H(x) = \log_2 M]$.

$H(x) = \log_2 2 = \underline{\underline{1}}$

$\therefore P(x_1) = P(x_2) = 1/2 \quad (1/M)$

The probability of the symbol $\{1/2, 1/2\}$ is equal to the capacity of the channel.

IW Q. The joint probability of a pair of random variable is given by $P[x,y] = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1/3 \end{bmatrix}$. Determine $H(X)$, $H[X,Y]$, $H[x|y]$, $I[x,y]$, $H[Y/x]$. Verify the relationship between joint, marginal and conditional entropy.

Sol: $P[x,y] = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1/3 \end{bmatrix}$

$P[x] = \{P(x_1), P(x_2)\} = \{2/3 \;\; 1/3\}$

$P[y] = \{P(y_1), P(y_2)\} = \{1/3 \;\; 2/3\}$

$\therefore H[x] = \sum_k P(x_k) \log_2 \frac{1}{P(x_k)}$

$= \frac{2}{3} \log_2 \frac{1}{2/3} + \frac{1}{3} \log_2 \frac{1}{1/3}$ $= 0.3899[illegible] +$ $0.52[illegible]206$

$H(x) = 0.9182$ bits/Symbol.

$H[y] = \sum_j P(y_j) \log_2 \frac{1}{P(y_j)}$

$= \frac{1}{3} \log_2 \frac{1}{1/3} + \frac{2}{3} \log_2 \frac{1}{2/3}$

$= 0.9182$ bits/Symbol.

$H[x, y] = \sum_k \sum_j P(x_k\, y_j) \log_2 \frac{1}{P(x_k\, y_j)}$

$= \left[\frac{1}{3} \log_2 \frac{1}{1/3}\right] \times 3 + 0$

$= 1.5849$ bits/Symbol.

$H(x/y) = \sum_k \sum_j P(x_k\, y_j) \log \frac{1}{P(x_k/y_j)}$

$P(x_k/y_j) = \frac{P(x_k\, y_j)}{P(y)}$

$= \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$

$\therefore H(x/y) = \frac{1}{3} \log_2 \frac{1}{1} + \frac{1}{3} \log_2 \frac{1}{1/2} + 0 +$

$\frac{1}{3} \log_2 \frac{1}{1/2}$

$= 2/3 = 0.6666$ bits/Symbol.

$H(y/x) = \sum_k \sum_j P(x_k\, y_j) \log_2 \frac{1}{P(y_j/x_k)}$

$P(y/x_k) = \frac{P(x_k\, y_j)}{P(x_j)} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

$\therefore H(y/x) = \frac{1}{3} \log_2 \frac{1}{1/2} + \frac{1}{3} \log_2 \frac{1}{1/2} + 0 + \frac{1}{3} \log_2 \frac{1}{1}$

$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

$= 0.6666$ bits/Symbol

$$I(x,y) = H(x) - H(x|y)$$
$$= 0.9182 - 0.6666$$
$$= \underline{0.2516 \text{ bit}}.$$

* $H[x,y] = H[y] + H[x|y]$

$\cancel{0.666+} \Rightarrow 1.5849 \neq 0.9182 + 0.6666$

$= 1.5849 = 1.5848$ bits/Symbol.

* $H[x,y] = H[x] + H[y|x]$

$1.5849 = 0.9182 + 0.6666 = \underline{1.5848}$ bits/Symbol.

* $H[x] \geq H[x|y] \Rightarrow 0.9182 \geq \underline{0.6666}$

* $H[y] \geq H[y|x] \Rightarrow 0.9182 \geq \underline{0.6666}$

* $H[x,y] = H[x] + H[y]$

$1.5849 = 0.9182 + 0.9182$

$\simeq \underline{1.8364}$.

-12/24 **Shanon channel coding Theorem:**

Shanon theorem on channel capacity [+Ve Stmt]:

Given a source of M equally likely messages with M>>1 which is generating an information at an information rate R and a channel capacity C; if $R \leq C$ then there exists a coding technique such that o/p of the source may be transmitted with a probability of error receiving messages that can be made arbitrary small.

$$\boxed{\frac{H[S]}{T_s} \leq \frac{C}{T_s}}$$

Where $H[S]$ is the entropy produced in every T_s Second. Let DM channel has a capacity C and used once every T_c Second.

Negative Statement:

Given the Source M equally likely message where $M >> 1$ which is generating information at information rate R and channel capacity C. If $R > C$ then the probability of error receiving the message at T_s Second are close to unity for every Set of M transmitted Signal.

$$\boxed{\frac{H[S]}{T_s} > \frac{C}{T_c}}$$

Channel coding.

Inorder to achieve high level of reliability while communicating through noisy channel, we add certain number of redundant bits with message bits and the process is known as channel coding/ error control coding.

Channel encoder adds the redundant bits with the original message and the channel decoder make use the redundant bit to detect and correct the errors. A Source encoder reduces the number of bits inorder to improve the efficiency of communication whereas the channel enco introduces extra bit to improve the reliability and Security of the transmission.

Code rate is defined as the ratio of number of message bit to the codeword length

ie; Code rate $\boxed{r = \frac{k}{N}}$

k = no. of message bit

N = Codeword length. $\underline{N-k}$ is the parity bit

imp.

Shanon-Hartley law:

Channel capacity Bandlimited Gaussian noise

It gives relationship between channel capacity and bandwidth.

$$C = B\log_2\left(1 + \frac{S}{N}\right) \text{ bits/Seconds}$$

C → Capacity of a band limited Gaussian channel with AWGN (Additive White Gaussian Noise)

B → Channel Bandwidth in Hz

S → Signal power in watts

N → Noise power in watts = ηB

$\frac{\eta}{2}$ → power Spectral density of white Gaussian.

S → Signal power

N → Noise power

Net power = S+N

Net voltage level = $\sqrt{S+N}$

Voltage level of noise = $\sqrt{N}$

We have $\frac{\sqrt{S+N}}{\sqrt{N}}$ distinguishable voltage levels = M

Each level (Quantisation level) represent each message i.e., number of messages = M

Maximum entropy $H_{max} = \log_2 M$

$$H_{max} = \log_2 \left[\frac{\sqrt{S+N}}{\sqrt{N}}\right] \text{ bits/message}$$

Let the channel be limited to $B\,Hz$ (Bandwidth)

Each sampling produces messages thus sample rate is same message rate or ~~f~~ 'r'

If channel is sampled at nyquist rate

$f_s = r = 2B$

Information rate $R = rH$

$H = H_{max}$

$$R_{max} = 2B \log_2 \left(\frac{S+N}{N}\right)^{1/2}$$

$$= \frac{1}{2} \times 2B \log_2 \left(\frac{S+N}{N}\right)$$

$$R_{max} = B \log_2 (1 + S/N)$$

Channel capacity is the maximum information rate a channel can transmit

$$\boxed{R_{max} = C = B \log_2 (1 + S/N)} \text{ bit/second.}$$

26/2/24

Q) A Voice grade channel of network has a bandwidth of 2.4 kHz. Calculate the information capacity of the channel for SNR of 20 dB. Calculate the minimum signal to noise ratio required to support information through channel at the rate 9.6 kilo bytes/second

Sol: Given

Bandwidth = 2.4 KHz

$= 2.4 \times 10^3$ Hz.

$$SNR = \frac{S}{N} = 20\ db$$

$$10 \log_{10}\left(\frac{S}{N}\right) = \cancel{20\ db}\ 20\ db$$

$$\frac{S}{N} = \cancel{\frac{1}{10\log_{10}}\ 20} = \text{Antilog}\left(\frac{20}{10}\right) = \text{Antilog}(2)$$

$$\frac{S}{N} = 100$$

$$C = 2.4 \times 10^3 \log_2 (1 + 100)$$

$$= 6.64438 \times 2.4 \times 10^3$$

$$= 15.94 \times 10^3 \text{ bits/Second.}$$

$$C = B \log_2\left(1 + \frac{S}{N}\right)$$

$$\log_2\left(1 + \frac{S}{N}\right) = \frac{C}{B}$$

$$\frac{S}{N} = \left[2^{\frac{C}{B}} - 1\right] = \left[2^{\frac{9.6\times10^3}{2.4\times10^3}} - 1\right] = 15$$

$$SNR = 10 \log_{10}\left(\frac{S}{N}\right)$$

$$= 11.76\ db.$$

Q) A Voice grade channel of a telephone n/w has a bandwidth of 3.4 kHz. Calculate the information capacity or channel capacity for SNR = 30db and also calculate the minimum SNR required to support the information transmit through the channel 4800 bits/second.

Sol: Bandwidth = 3.4 KHz

$= 3.4 \times 10^3$ Hz.

$SNR = 30\,db$

$10 \log_{10}(SNR) = 30$

$\frac{S}{N} = \text{Antilog } \frac{30}{10}$

$= \underline{\underline{1000}}$

$C = B \log_2(1 + S/N)$

$= 3.4 \times 10^3 \log_2(1 + 1000)$

$= \underline{\underline{33.883 \times 10^3}}$ bits/Seconds.

$C = B \log_2(1 + S/N)$

$\log_2(1 + S/N) = \frac{C}{B}$

$\frac{S}{N} = (2^{C/B} - 1) = \left[2^{\frac{4800}{3.4 \times 10^3}} - 1\right]$

$= \underline{\underline{1.6606}}$

$SNR = 10 \log_2\left(\frac{S}{N}\right)$

$= \underline{\underline{3.202\,db}}$

Implications of Shannon-Hartely Theorem

Shannon-Hartley theorem States that the channel capacity of a White bandlimited Gaussian channel is $C = B \log_2(1 + \frac{S}{N})$ b

where B is the channel bandwidth

S is the average Signal power

N is the average Noise power

Q A Gaussian channel has 1 MHz bandwidth. Calculate the channel capacity if the Signal power to noise Spectral density ratio is 10^5 Hz. Also find the maximum information rate.

Sol: Given,

$W = B = 1\,MHz = 1 \times 10^6\,Hz$

$C = B \log_2 (1 + S/N)$

$$\boxed{C = W \log_2 (1 + S/WN_0)}$$

$= 10^6 \log_2 \left(1 + \frac{10^5}{10^6}\right)$

$= 137.50 \times 10^3$ bits/Second

$C = 137503.5$ bits/Second.

$$\boxed{R = 1.44 \frac{S}{N_0}}$$

$= 1.44 \times 10^5$

$R = 144000$ bits/Second.

Q A Gaussian channel has a 10MHz bandwidth. If the Signal to noise ratio is 100 calculate the channel capacity and maximum information Rate.

Sol: $W = 10 \times 10^6\,Hz$

$C = W \log_2 \left(1 + \frac{S}{N_0 W}\right)$

$= 10 \times 10^6 \log_2 \left(1 + \frac{100}{10^7}\right)$

$= 144.268$ bits/Second.

$R = 1.44 \frac{S}{N_0} = 1.44 \times 100$

$= 144$ bits/Second.

Q. A White grade channel of a telephone n/w has a bandwidth of 3.4 kHz. Determine the information capacity of a telephonic channel for a SNR of 30 db.

Sol: Given:

$$B = 3.4 \times 10^3 \text{ Hz}$$

$$SNR = 30 \text{ db}$$

$$10 \log_{10}(SNR) = 30$$

$$\frac{S}{N} = \text{Antilog } \frac{30}{10}$$

$$= \underline{\underline{1000}}$$

$$C = B \log_2 (1 + S/N)$$

$$= 3.4 \times 10^3 \log_2 (1001)$$

$$= \underline{\underline{33.888 \times 10^3}} \text{ bits/Symbol.}$$

Q. A channel has a bandwidth of 8 kHz. What is the channel capacity if SNR is 31 for the same channel capacity if the SNR is 61. What be the new channel bandwidth.

Sol: $B = 8 \times 10^3$ Hz

$$C = B \log_2 \left(1 + \frac{S}{N_0}\right)$$

$$= 8 \times 10^3 \log_2 \left(1 + 31\right)$$

$$C = \underline{\underline{40 \times 10^3}}$$

$$40 \times 10^3 = B \log_2 (62)$$

$$40 \times 10^3 = 5.9541\, B$$

$$\therefore B = \underline{\underline{6.717 \text{ kHz}}}.$$

Differential Entropy

For a continous random Variable 'x' with a probability density function $f(x)$ the differential entropy is defined as

$$H[X] = -\int_S f(x) \log_2 f(x)\, dx$$

Ex: Uniform distribution.

If a random Variable x is distributed over a uniform distribution $f(x) = \frac{1}{a}$ over $0 \leq x \leq a$ then differential entropy of x

$$H(x) = -\int_0^a \frac{1}{a} \log_2 (1/a)\, dx$$

$$= -\frac{1}{a} \int_0^a \log_2 (1/a)\, dx$$

$$= -\frac{1}{a} \log_2 \frac{1}{a} (x)_0^a$$

$$= -\frac{1}{a} \times a \log_2 \frac{1}{a} \cancel{(a)}$$

$$= -\log \frac{1}{a} = \log_2 a \text{ bits/Symbol}$$

If $a=1$ $\log_2(1) = 0$

$a=2$ $\log_2(2) = 1$

$a=1/2$ $\log_2(1/2) = -1$

The average mutual information in a continuous channel is defined by analog with discrete case as

$$I[x,y] = H[x] - H[x|y]$$

$$= H[y] - H[y|x]$$

$$H[y] = \int f(y) \log_2 f(y)\, dy$$

$H[x] = -\int f(x) \log_2 f(x)\, dx$

$H[x|y] = \iint f(x,y) \log_2 f(x|y)\, dx\, dy$

$H[y|x] = \iint f(x,y) \log_2 f(y|x)\, dx\, dy$

Differential entropy of a Gaussian random variable differential entropy

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Where σ is standard deviation if $\mu = 0$ then

$$H(x) = -\int_{-\infty}^{\infty} f(x) \log f(x)\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

$$H(x) = -\int_{-\infty}^{\infty} f(x) \log f(x)\, dx$$

$$= -\int_{-\infty}^{\infty} f(x) \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}\right] dx$$

$$= -\int_{-\infty}^{\infty} f(x) \log \frac{1}{\sqrt{2\pi\sigma^2}}\, dx - \int_{-\infty}^{\infty} f(x) \log e^{-x^2/2\sigma^2}\, dx$$

$$= \log\sqrt{2\pi\sigma^2} \int_{-\infty}^{\infty} f(x)\, dx - \log e \int_{-\infty}^{\infty} f(x) \frac{-x^2}{2\sigma^2}\, dx$$

$$= \log\sqrt{2\pi\sigma^2} \underbrace{\int_{-\infty}^{\infty} f(x)\, dx}_{1} + \frac{\log e}{2\sigma^2} \underbrace{\int_{-\infty}^{\infty} x^2 f(x)\, dx}_{\sigma^2}$$

$$= \log\sqrt{2\pi\sigma^2} + \frac{\log e}{2\sigma^2} \cdot \sigma^2$$

$$= \log\sqrt{2\pi\sigma^2} + \frac{1}{2}\log e$$

$$= \log\sqrt{2\pi\sigma^2} + \log\sqrt{e}$$

$$H(x) = \log\sqrt{2\pi e\sigma^2} \text{ bits/message}$$

Third Implication:

From Shanon-Hartley theorem,

$$C = W \log \left[1 + \frac{S}{W N_0}\right] \text{ bits/Second.}$$

A noiseless channel ($N = 0$; $S/N = \infty$), have an infinite capacity, but channel capacity doesnot become infinite as bandwidth approaches infinity, because with an increase in bandwidth, the noise power also increases.

For a fixed signal power, in the presence of white gaussian noise, the channel capacity approches an upperlimit with increasing bandwidth. When 'W' increased, C also increases and $R_{max} = C$, the maximum rate of information transmission.

$$C = W \log \left[1 + \frac{S}{W N_0}\right]$$

$$= \frac{S}{N_0} \left\{ \frac{W \cdot N_0}{S} \log \left[1 + \frac{S}{W \cdot N_0}\right] \right\}$$

$$= \frac{S}{N_0} \log \left[1 + \frac{S}{W \cdot N_0}\right]^{\frac{W N_0}{S}}$$

Let $x = \frac{S}{W \cdot N_0}$

Then $C = \frac{S}{N_0} \log (1+x)^{1/x}$

When $W \to \infty$; $x \to 0$

$$\lim_{x \to 0} [1+x]^{1/x} = e$$

Then

$$\lim_{W \to \infty} C = \lim_{W \to \infty} \frac{S}{N_0} \log (1+x)^{1/x}$$

$$\lim_{W \to \infty} C = \frac{S}{N_0} \lim_{x \to 0} \log (1+x)^{1/x}$$

$$= \frac{S}{N_0} \log_2 \left\{ \lim_{x \to 0} (1+x)^{1/x} \right\}$$

Also

$$\boxed{\begin{aligned} C_\infty &= \frac{S}{N_0} \log_2 e \text{ bits/sec} \\ C_\infty &= W \cdot \frac{S}{N_0} \log_2 e \text{ bits/sec} \\ &= 1.44 W \frac{S}{N} \text{ bits/sec} \\ &= R_{max} \end{aligned}}$$

Shanon's Limit:

$E_b \rightarrow$ Average transmitted energy per information bit.

$\frac{E_b}{N_0} \rightarrow$ SNR

$\frac{C}{W} \rightarrow$ Bandwidth - efficiency

$\frac{E_b}{N_0} \geq -1.59 \text{ dB}$

The Value -1.6 dB is known as fundemental Shanon limit. If SNR is less than this limit it is not possible to reach error possibility that tends to zero.

Trade-off between SNR and Bandwidth:

(2nd Implication)

An important implication of Shanon-Hartley law is the exchange of bandwidth with SNR & Vice-Versa.

Let $\frac{S_1}{N_1} = 15$ and $w_1 = 5\,kHz$

$$C = w_1 \log_2 \left(1 + \frac{S_1}{N_1 W_1}\right)$$

$$= 5\times10^3 \log_2 \left(1 + \frac{15}{5\times10^3}\right)$$

$$= 21.6080 \text{ bits/Second.}$$

Keeping the channel capacity $C_2 = C_1$ and the SNR is increased to 31 $\left[\frac{S_2}{N_2} = 31\right]$,

$$21.6080 = W_2 \log_2 \left(1 + 31\right)$$

$$\therefore W_2 = 4.361\,kHz.$$

Since the noise power $N = W N_0$ as the bandwidth get reduced from 5 kHz to 4 kHz the noise also decrease indicating an increase in the signal power.

$$N_1 = W_1 N_0$$

$$= 5\times10^3 N_0 \quad \text{——} \quad (1)$$

$$N_2 = W_2 N_0$$

$$= 4\times10^3 N_0 \quad \text{——} \quad (2)$$

$$\frac{\frac{S_2}{N_2}}{\frac{S_1}{N_1}} = \frac{31}{15} \quad \text{——} \quad (3)$$

From the equation (1) (2) and (3):

$$\frac{S_2}{S_1} = \frac{31 N_2}{15 N_1} = \frac{31\times4\times10^3 \cancel{N_0}}{15\times5\times10^3 \cancel{N_0}} = \frac{124000}{75000}$$

$$\frac{S_2}{S_1} = 1.653$$

With a ~~4 KHz~~ bandwidth, ~~the noise~~ po
~~it 0.~~

To decrease the bandwidth, the signal power has to be increased. Similarly decreasing signal power increases bandw

8/2/24

MODULE - 3

Overview of Groups, Rings & finite field

Basic Properties:

(A_1) Closure : If a and b belong to G, then $a*b$ is also in G.

(A_2) Associative: $a*(b \cdot c) = (a \cdot b)*c$ for all a, b, c in G

(A_3) Identity element : There is an element e in G such that $a*e = e*a = a$ for all a in G

$[a \cdot e = e \cdot a = a]$

(A_4) Inverse element : For each a in G, there is an element a' in G, Such that $a*a' = a'*a = e$

$2 \times \frac{1}{2} = \frac{1}{2} \times 2 = 1$

(A_5) Commutative: $a \cdot b = b \cdot a$ for all a, b in G

(M_1) Closure under multiplications : for $a, b, c \in R$

$a \times b \in R$

(M_2) Associtivity of multiplication for $a, b, c \in R$

$a \times \cancel{b} (b \times c) = (a \times b) \times c$

(M_3) Distributive law : for $a, b, c \in R$

$a \times (b + c) = a \times b + a \times c$

$(a + b) \times c = a \times c + b \times c$

(M_4) Commutativity of Multiplication : $ab = ba$ for all a, b in R.

(M_5) Multiplicative identity : There is an element 1 in R Such that $a1 = 1a = a \ \forall a$ in R

(M6) No zero divisors: If a, b in R and ab = 0
then either a = 0 or b = 0

(M7) Multiplicative inverse: for each a in F, except 0,
there is an element a^{-1} in F such that
$a(a^{-1}) = (a^{-1})a = 1$

Group

A group is a set of symbols with a law * defined on it. A group (G, *) satisfies following axioms

1. Closure
2. Associativity
3. Identity
4. Inverse

Rings:

A ring is a set of symbols with two laws (+, x) defined on it, A ring (R, +, x) satisfies the following axioms

$A_1 - A_5$ (Abelian group/ commutative group)

$M_1 - M_3$ & A

field:

field (F, +, x) Satisfies

$A_1 - A_5$

$M_1 - M_4$

M_5, M_6, M_7

Field

Ring

Abelian Group

Group

- A_1 Closure under addition
- A_2 Assotivity of addition
- A_3 Additive Identity
- A_4 Additive Inverse
- A_5 Commutativity of addition
- M_1 Closure under multiplication
- M_2 Assotivity of multiplication
- M_3 Distributive law
- M_4 Commutativity of multiplication
- M_5 Multiplication Identity
- M_6 No zero divisors
- M_7 Multiplicative Inverse.

Block code and it's parameters:

Error correcting and detecting code

Eg: block code.

Block codes are a large and important family of error correcting code that encode data in blocks. Block code represent (n, k)

$n \rightarrow$ Code word length

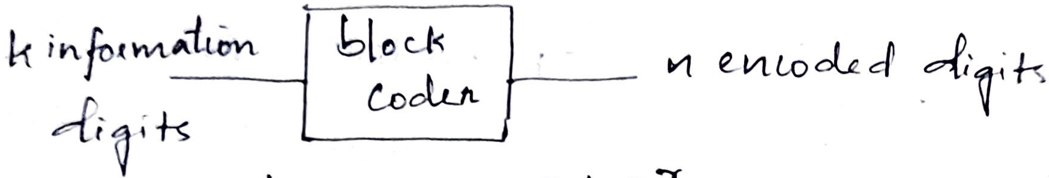
$k \rightarrow$ message length

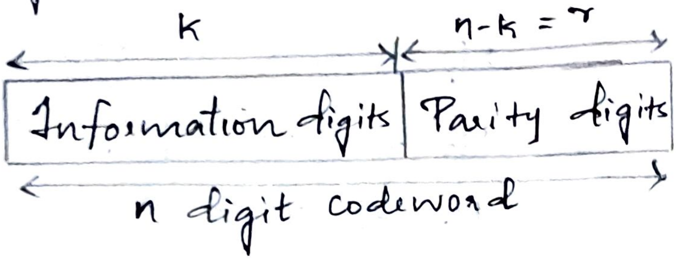
$n-k = r \rightarrow$ redundant bit

Block codes: Block codes consists of $(n-k)$ number of check bits (redundant bits) being added to ie; number of information bits to form 'n' bit code words.

These $(n-k)$ number of check bits are "derived from k information bits". At the receiving

Check bits are used to detect and corr[ect]
error which may occur in the entire
bit code words.





Properties:

1. Message bit length (k)
2. Redundant bit length (r)
3. Codeword or block length (n)
4. The rate R = the rate of a block code is defined as the ratio between its message length and its block length.

$$R = \frac{k}{n}$$

Q For a hamming distance of 5 how ma[ny] error can be detected and how many errors can be corrected.

Sol: $d_{min} = 5$

No. of errors that can be dectected

$= s+1 \le d_{min}$

$s \leq d_{min} - 1$

$s \leq 4$

At least 4 errors can be detected.

Error Correction $\quad 2t + 1 \leq d_{min}$

$2t \leq 5 - 1$

$2t \leq 4$

$t \leq 2$

No. of errors that can be corrected $= \underline{\underline{2}}$

Q The parity equation of a linear block code are $c_5 = d_1 \oplus d_3 \oplus d_4$, $c_6 = d_1 \oplus d_2 \oplus d_3$, $c_7 = d_2 \oplus d_3 \oplus d_4$ where d_1, d_2, d_3, d_4 are the message bits and $c_5\ c_6\ c_7$ are parity bits. Find

a) Generate a matrix G

b) The parity check matrix H

c) Draw the encoder circuits

d) Write all the possible code word

e) Minimum distance code word

f) How many erros can be detected & corrected

g) Draw the decoder circuit.

h) Illustrate the decoding

Sol: Given,

$c_5 = d_1 \oplus d_3 \oplus d_4$

$c_6 = d_1 \oplus d_2 \oplus d_3$

$c_7 = d_2 \oplus d_3 \oplus d_4$

Parity Matrix, $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

a) $G = [I : P]$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

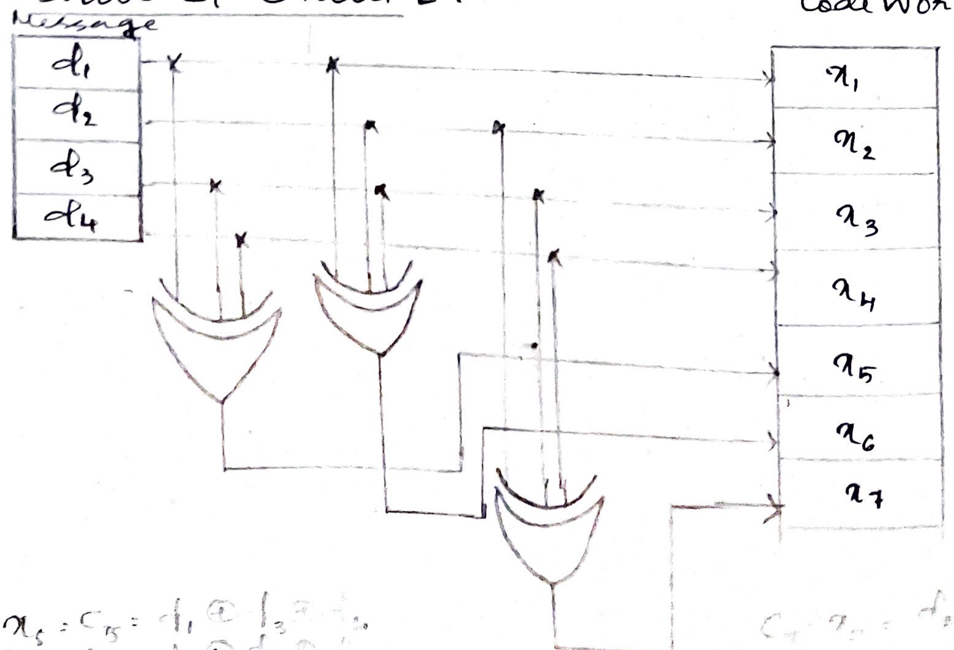
$$\therefore G = \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

b) The parity check matrix $H = [P^T : I]$

$$P^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore H = \left[\begin{array}{cccc:ccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

c) Encoder Circuit:



$x_5 = c_5 = d_1 \oplus d_3 \oplus$

d) To generate the possible codeword, the minimum codeword $= 2^k$.

$k = 4 \Rightarrow 2^4 = \underline{16}$

Message bit				Codeword			Code weight
d_1	d_2	d_3	d_4	C_5	C_6	C_7	
0	0	0	0	0	0	0	0
0	0	0	1	1	0	1	3
0	0	1	0	1	1	1	4
0	0	1	1	0	1	0	3
0	1	0	0	0	1	1	3
0	1	0	1	1	1	0	4
0	1	1	0	1	0	0	3
0	1	1	1	0	0	1	4
1	0	0	0	1	1	0	3
1	0	0	1	0	1	1	4
1	0	1	0	0	0	1	3
1	0	1	1	1	0	0	4
1	1	0	0	1	0	1	4
1	1	0	1	0	0	0	3
1	1	1	0	0	1	0	4
1	1	1	1	1	1	1	7

e) Minimum distance of codeword = 3

f) Detection:

$s + 1 \le d_{min}$

$s + 1 \le 3$

$\underline{s \le 2}$

Correction:

$2t + 1 \le d_{min}$

$2t \le 2$

$\underline{t \le 1.}$

Two errors can be detected and one error can be corrected.

2) Decoder [Syndrome Decoding]

$$\boxed{S = Y H^T}$$

$Y = y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7$ is the received code word.

$$H^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{[S_1\ S_2\ S_3]}_{\text{no of parity}} = [y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7]\, H^T$$

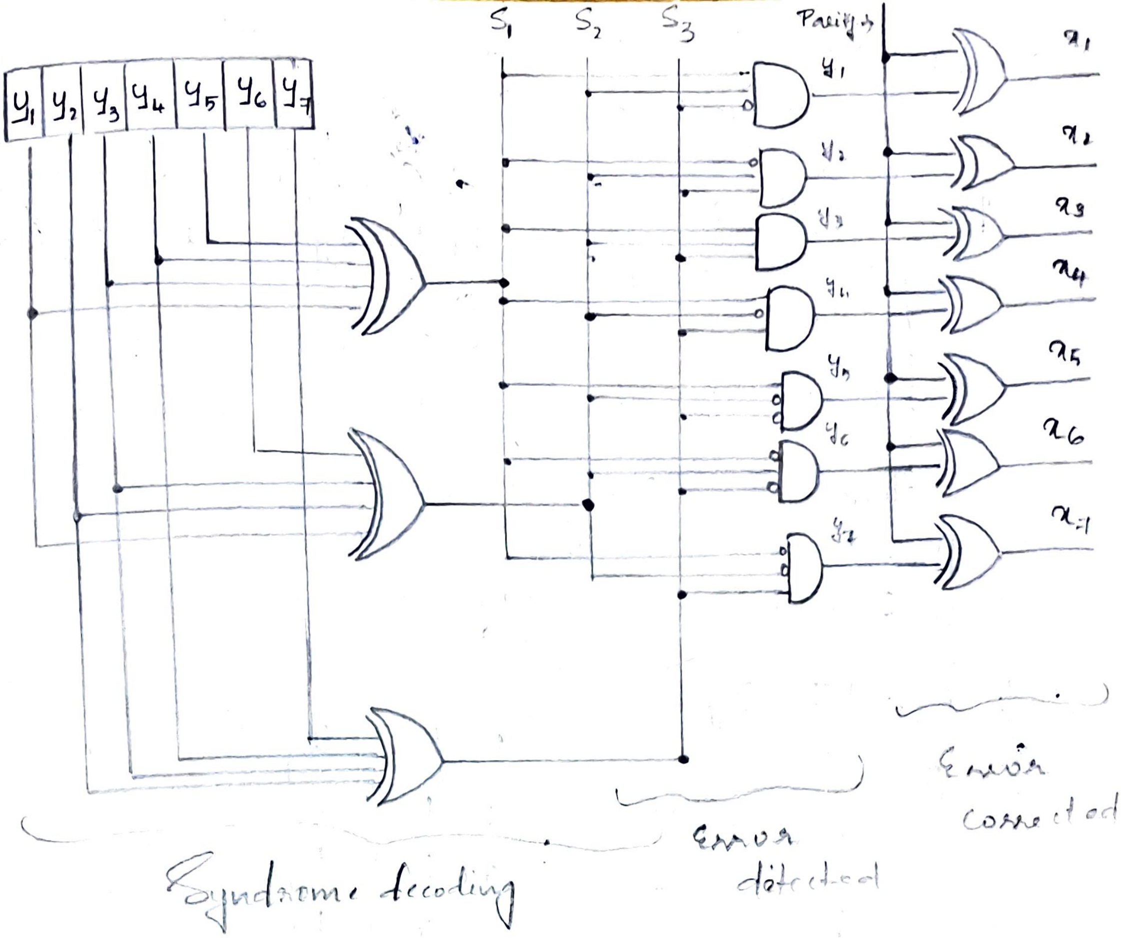
$$= [y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1 0 1 1.

$S_1 = y_1 \oplus y_3 \oplus y_4 \oplus y_5$

$S_2 = y_1 \oplus y_2 \oplus y_3 \oplus y_6$

$S_3 = y_2 \oplus y_3 \oplus y_4 \oplus y_7$



h) Assume the received codeword $y = 1001111$

$S = rH^T$

$C = \cancel{10110}$

$S = \underset{1\times n}{Y} H^T_n$

$S_1 = 1 \qquad S_2 = 0 \qquad S_3 = 0$

$\therefore S = [1\ 0\ 0]$

bit position of H^T

The error is detected in the ~~4~~ 5 bit.

Error correction:

1001111

$$
\begin{array}{r}
y = 1\,0\,0\,1\,1\,1\,1 \oplus \\
0\,0\,0\,0\,1\,0\,0 \\
\hline
1\,0\,0\,1\,0\,1\,1
\end{array}
$$

Corrected code word = 1001011

28/2/24 Q $C_5 = d_1 \oplus d_2 \oplus d_4$ $\quad C_7 = d_1 \oplus d_3 \oplus d_4$

$C_6 = d_1 \oplus d_2 \oplus d_3$ $\quad C_8 = d_2 \oplus d_3 \oplus d_4$

$$P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

a) Generate matrix G
b) Parity check matrix H
c) Encoder circuit
d) Possible code word
e) Minimum distance
f) How many errors can be corrected & detected
g) Decoder circuit
h) Illustrate decoding.

Sol: $G = [I : P]$

a)
$$G = \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right]$$

b) $H = [P^T : I]$

$$P^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$H = \left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

c) Encoder circuit.

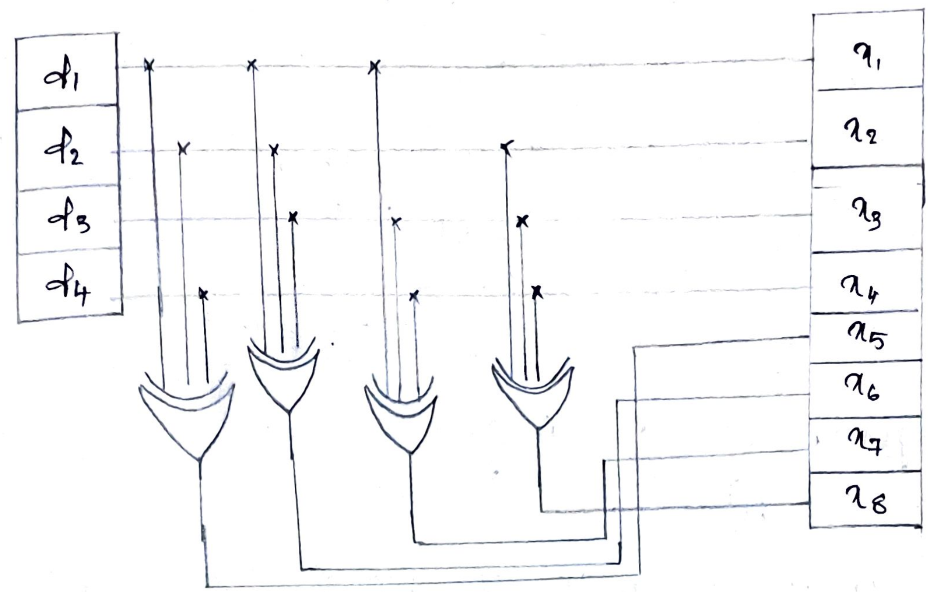
$C_5 = d_1 \oplus d_2 \oplus d_4$

$C_6 = d_1 \oplus d_2 \oplus d_3$

$C_7 = d_1 \oplus d_3 \oplus d_4$

$C_8 = d_2 \oplus d_3 \oplus d_4$

Message Codeword.



d) To generate the possible codeword the minimum codeword $= 2^k = 2^4 = \underline{16}$

Message d_1	d_2	d_3	d_4	Code word c_5	c_6	c_7	c_8	code weight
0	0	0	0	0	0	0	0	0
0	0	0	1	1	0	1	1	4
0	0	1	0	0	1	1	1	4
0	0	1	1	1	1	0	0	4
0	1	0	0	1	1	1	1	5
0	1	0	1	0	1	1	0	4
0	1	1	0	1	0	1	0	4
0	1	1	1	0	0	0	1	4
1	0	0	0	1	1	1	0	4
1	0	0	1	0	1	0	1	4
1	0	1	0	1	0	0	1	4
1	0	1	1	0	0	1	0	4
1	1	0	0	0	0	1	1	4
1	1	0	1	1	0	0	0	4
1	1	1	0	0	1	0	0	4
1	1	1	1	1	1	1	1	8

e) $d_{min} = \underline{4}$

f) Error detected, $s+1 \le d_{min}$

$$s+1 \le 4$$
$$s \le \underline{3}$$

Error corrected, $2t+1 \le d_{min}$

$$2t+1 \le 4$$
$$2t \le 3$$
$$t \le 3/2 = \underline{1.5}$$

g) <u>Syndrome decoding.</u>

$$S = YH^T$$

$y = y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7\ y_8$ received codeword

$$YH^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{bmatrix}$$

$S_1 = y_1 \oplus y_2 \oplus y_4 \oplus y_5$ $\cancel{1 \oplus 1 \oplus 1 \oplus 1}$

$S_2 = y_1 \oplus y_2 \oplus y_3 \oplus y_6$

$S_3 = y_1 \oplus y_3 \oplus y_4 \oplus y_7$ $\quad 1 \oplus 0 \oplus 1$

$S_4 = y_2 \oplus y_3 \oplus y_4 \oplus y_8$

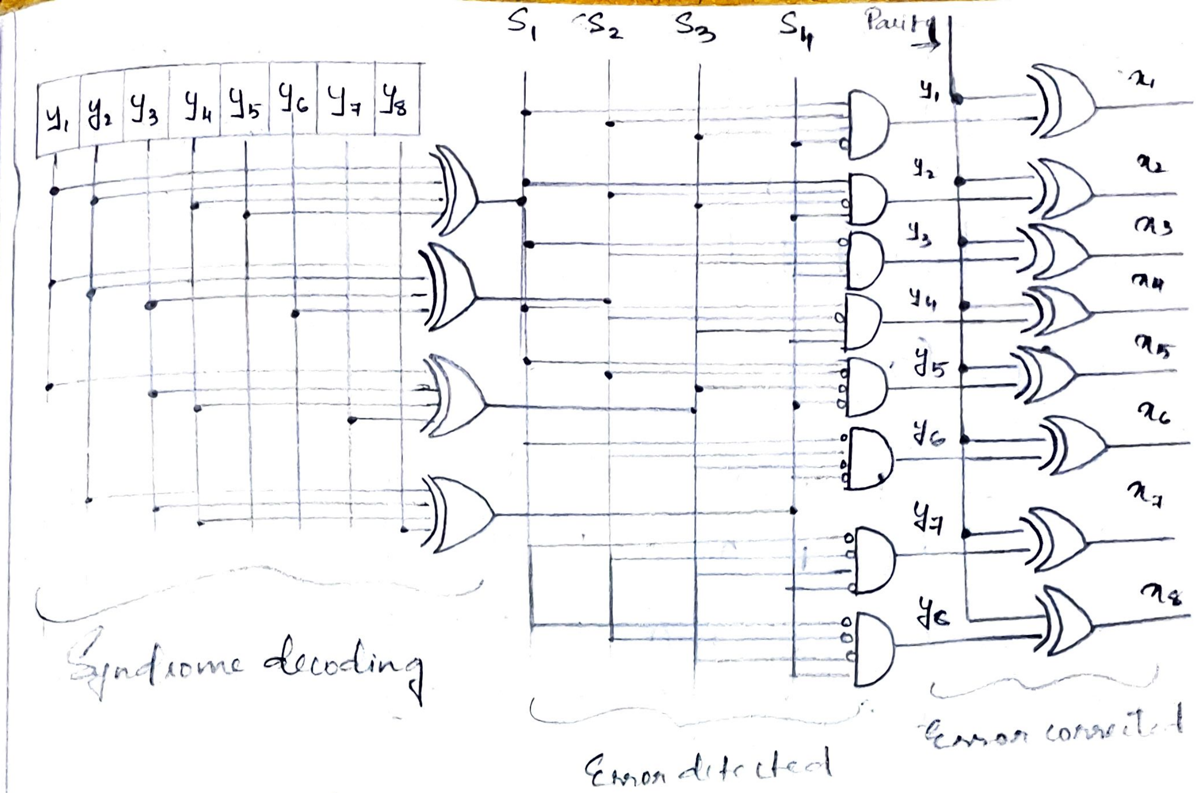
~~Assume~~

$$\begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{array}$$
$$y'' = 10011101$$

$S_1 = 1, \quad S_2 = 0 \quad S_3 = 0 \quad S_4 = 0$

$$S = \underline{[1000]}$$

The error is detected in the ~~8~~ 5^{th} bit



Error correction.

$$
\begin{array}{rl}
y = & 1\,0\,0\,1\,1\,1\,0\,1 \oplus \\
 & 0\,0\,0\,0\,1\,0\,0\,0 \\
\hline
 & 1\,0\,0\,1\,0\,1\,0\,1
\end{array}
$$

Corrected code word = 10010101

24 Q The parity matrix of a $7,4 linear block code is given as

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Generate a parity check matrix G, write the parity matrix H. Write the parity equation. Draw the encoder diagram. Generate all the possible code word. Find the minimum distance of the code. Write the error correction and detection possibility. Draw the decoder circuit. Illustrate the decoding if the received code word is 1000111, 1010101

Sol: $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$C_5 = d_1 \oplus d_2 \oplus$
$C_6 = d_1 \oplus d_3 \oplus$
$C_7 = d_1 \oplus d_2 \oplus$

2023

12 (b)

$G = [I : P]$

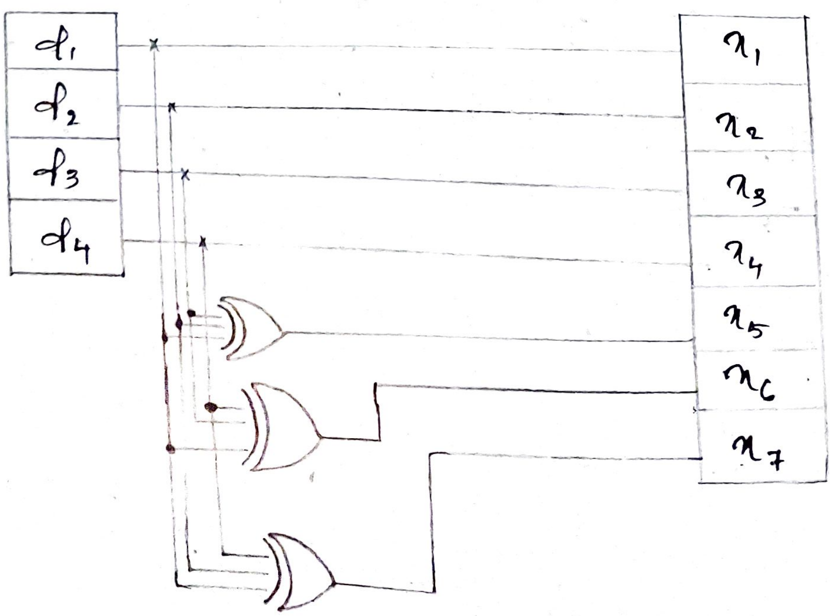
$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

$H = [P^T : I]$

$= \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Encoder circuit:

Messages . Codeword.



The minimum no. of codeword $= 2^k = 2^4 = \underline{16}$

Messages				Code word			Weight
d_1	d_2	d_3	d_4	c_5	c_6	c_7	
0	0	0	0	0	0	0	0
0	0	0	1	0	1	1	3
0	0	1	0	1	1	0	3
0	0	1	1	1	0	1	4
0	1	0	0	1	0	1	3
0	1	0	1	1	1	0	4
0	1	1	0	0	1	1	4
0	1	1	1	0	0	0	3
1	0	0	0	1	1	1	4
1	0	0	1	1	0	0	3
1	0	1	0	0	0	1	3
1	0	1	1	0	1	0	4
1	1	0	0	0	1	0	3
1	1	0	1	0	0	1	4
1	1	1	0	1	0	0	4
1	1	1	1	1	1	1	7

$d_{min} = 3$

Error detected,

$s + 1 \le d_{min}$

$s + 1 \le 3$

$\underline{s \le 2}$

Error corrected,

$2t + 1 \le d_{min}$

$2t + 1 \le 3$

$2t \le 2$

$\underline{t \le 1}$

Syndrom decoding.

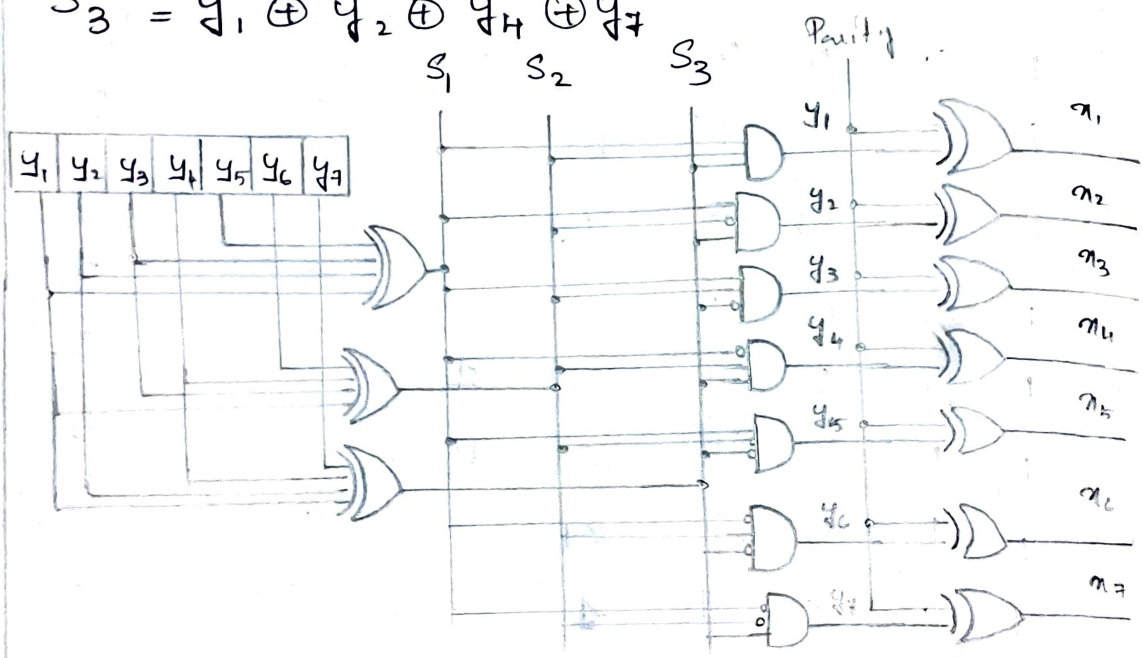
$$S = Y H^T$$

$$S = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_1 = y_1 \oplus y_2 \oplus y_3 \oplus y_5$$

$$S_2 = y_1 \oplus y_3 \oplus y_4 \oplus y_6$$

$$S_3 = y_1 \oplus y_2 \oplus y_4 \oplus y_7$$



The received code word $y = 1\ 0\ 0\ 0\ 1\ 1\ 1$

$$S = [0\ \ 0\ \ 0]$$

No error.

The received code word $y = 1\ 0\ 1\ 0\ 1\ 0\ 1$

$$S = [1\ \ 0\ \ 0]$$

The error is in the 5^{th} bit.

$$\begin{array}{r} 1\,0\,1\,0\,1\,0\,1 \\ \oplus\ 0\,0\,0\,0\,1\,0\,0 \\ \hline 1\,0\,1\,0\,0\,0\,1 \end{array}$$

The corrected code word = $\underline{1010001}$

Standard Array & Syndrome Calculation

/3/24

Q For a Systematic 6, 3 linear block code the parity matrix is given by,

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Construct the Standard array.

Sol: Given, $n = 6$

$k = 3$

When $k = 3$, the no. of messages $= 2^3 = \underline{8}$

Then, the message Vectors are,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The code Vectors are found using,

$$\boxed{[C] = [D][G]}$$

D = data

G = Generator matrix.

$$G = [I_K : P]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$[C] = [D][G]$

$$[C] = [d_1\ d_2\ d_3] \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$= [d_1\ \ d_2\ \ d_3\ \ d_1 \oplus d_3\ \ d_2 \oplus d_3\ \ d_1 \oplus d_2]$$

Coordinate	Message Vector	Code Vector
C_1	0 0 0	0 0 0 0 0 0
C_2	0 0 1	0 0 1 1 1 0
C_3	0 1 0	0 1 0 0 1 1
C_4	0 1 1	0 1 1 1 0 1
C_5	1 0 0	1 0 0 1 0 1
C_6	1 0 1	1 0 1 0 1 1
C_7	1 1 0	1 1 0 1 1 0
C_8	1 1 1	1 1 1 0 0 0

If the received vector is a) 100100
b) 110100

a) If the received Vector is 100100 then the corrected code word = 100101

b) If the received Vector is 110100 then the corrected code Word = 110110

Syndrome Value of H^T	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
s_1 s_2 s_3	000000	001110	010011	011101	$\boxed{100101}$	101011	$\boxed{110110}$	111000
101	100000	101110	110011	111101	000101	001011	010110	011000
011	010000	011110	000011	001101	110101	111011	000110	001000
110	001000	000110	011011	010101	101101	100011	111110	110000
100	000100	001010	010111	011001	100001	101111	110010	111100
010	000010	001100	010001	011111	100111	101001	110100	111010
001	000001	001111	010010	011100	100100	101000	110111	111001

$H =$

$d_1 \oplus d_3 \oplus d_4$

$d_1 \oplus d_2 \oplus d_4$

$d_2 \oplus d_3 \oplus d_4$

Syndrom calulation

$\boxed{S = R \times H^T}$ where R is the received code vector.

If the received code vector $R = 100100$

$$H^T = \begin{bmatrix} P \\ I_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

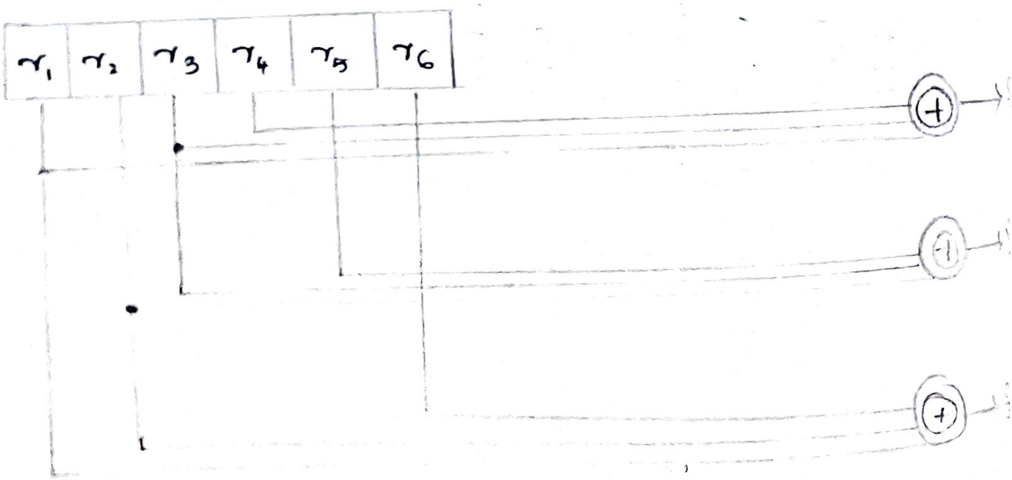
$S_1 = 1 \oplus 1 = \underline{\underline{0}}$ $\quad S_2 = 0 \quad S_3 = 1$

$S = [0 \;\; 0 \;\; 1]$

$S_1 = r_1 \oplus r_3 \oplus r_4$

$S_2 = r_2 \oplus r_3 \oplus r_5$

$S_3 = r_1 \oplus r_2 \oplus r_6$



~~For a~~

Standard array decoding.

For a n, k there are 2^{n-k} number of rows and $2^{k}+1$ number of coloumn. ~~A Standard~~

* First row of Standard array contain all the possible code word and this row is known as coset leader.
* The first coloumn contains the error pattern
* The elements in each coloumn are calculated by adding coset leader with error pattern.
* Decoding

If a received codeword is find in the standard array in any of the coloumn, then the coset leader will be the actual transmitted code word.

Q. Show that $C = \overset{x_1}{0000}, \overset{x_2}{1100}, \overset{x_3}{0011}, \overset{x_4}{1111}$ is ~~a~~ linear code. What is the minimum distance.

Sol: Linearity property:

The modulo two addition of any two valid code word gives another valid code word.

$x_1 \oplus x_2 \Rightarrow 0000 \oplus 1100 = \underline{\underline{1100}}$

$$\begin{array}{r} 0000 \\ 1100 \\ \hline 1100 \end{array}$$

Hence x_2 is Valid

$x_2 \oplus x_3 \rightarrow 1100 \oplus 0011 = \underline{\underline{1111}}$ Hence x_4 is Valid.

$$\begin{array}{r} 1100 \\ 0011 \\ \hline 1111 \end{array}$$

$x_2 \oplus x_4 \Rightarrow 1100 \oplus 1111 = \underline{\underline{0011}}$ Hence x_3 is Valid.

$$\begin{array}{r} 1100 \\ 1111 \\ \hline 0011 \end{array}$$

So C is a linear code.

The weight of the code are 0, 2, 2, 4

$\therefore$ Minimum weight 2 $\therefore d_{min} = \underline{\underline{2}}$

Detection and Correction of Error

For the detection & correction of transmission error, one or more extra b are added at the time of transmitting the signal. These extra bits are known as Redundant bit or Parity bit.

The parity bit detects and corrects the error. The data along the parity bit is known as Codeword.

Classification of codes.

The codes are basically classified as error detecting code and correcting code.

Error detection Technique.

1. Parity checking
2. Check ~~some~~ error detection
3. Cyclic redundancy check

Parity checking:

In the parity checking an additional bit called parity bit is added to the data/message. The additional bit is chosen that the resultant word either have even parity or odd parity.

If even parity is ~~used~~ used then, the parity bit is added to make the total no. of 1 bit ~~=~~ even.

Similarly, in odd parity the total no. of '1' bit is made odd by adding the parity.

Even parity-

Ex: data word - 1001011

the total no. of 1 ⇒ 4

So parity bit = 0

code word = 10010110

⇒ 0010011

the total no. of 1 ⇒ 3

So parity bit = 1

Code word = 00100111

Odd parity:

⇒ 1001011

the total no. of 1 = 4

So parity bit = 1

Code word = 10010111

1/3/23

Repitation Code [Basic error correcting code]

Inorder to transmit a message over a noisy channel that may corrupt the transmission in few places, the idea of repitation code is to just repeat the message several times.

The repeatation code are one of the few known codes whose code rate can be automatically adjusted to varying channel capacity by sending more or less parity info as required to overcome the channel noise and is also known for non-Erasure channel.

for $n, k = (3, 1)$

$n = 3$ $\quad r = n - k$

$k = 1$ $\quad = \underline{\underline{2}}$

Message bit. $\qquad$ Encoder.

0 $\qquad$ 000 redundant bits

or

1 $\qquad$ 111 "

* (4, 1)

$n = 4 \quad k = 1$

$r = 3$

Information bit	Parity bit	Codeword (Encoded)
0	000	0000
1	111	1111

Decoding: this doesnot have a effect on code as we have a maximum 1

110 000 111 → Error free code.

111 000 010 → Error code.

Hamming Code

Q Design a n, k hamming code with a minimum distance $d_{min} = 3$ and a message length of 4 bit.

Sol: $k = 4$.

The code length $n \leq 2^{n-k} - 1$

n is found using trial & error method,

$n = 1 \rightarrow 2^{-3} - 1 \quad \because n \nleq 2^{n-k} - 1$

$n = 4 \rightarrow 2^{0} - 1 = 3$

$n = 7 \rightarrow 2^{3} - 1 = \underline{7}$ $\qquad n \leq 7$

$7 \leq 7$

Hamming code equation $H^T = \begin{bmatrix} P \\ \hline I_{n-k} \end{bmatrix}$

$$H^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P =$

000
~~001~~
~~010~~
011 ✓
~~100~~
101 ✓
110 ✓
111 ✓

I_{n-k}

100
010
001

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 1 & & & 1 & 1 & \end{array}$$

$d_{min} = 4$

$$\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & & 1 & & 1 & & 1 \end{array}$$

$d_{min} = 4$

$T = 3/2$

Hamming Code.

These are linear block code that can correct single bit error for every integer there exist the code word length $n = 2^p - 1$. The no. of message bit $k = 2^p - p - 1$

No. of Parity bit $P = n - k$

The error correcting capability $T = \frac{d_{min} - 1}{2}$

5/3/24 Module - 4

Cyclic Code.

A (n, k) linear block code C is said to be cyclic code if every cyclic shift of the code is also a code vector of C

Ex: $C_1 = 0111001$ be a code vector of C

$C_2 = 1011100$ (last 1 of C_1 has moved int first position)

$C_3 = 0101110$ ∴

$C_4 = 0010111$

$V = (V_0, V_1, V_2 \dots V_{n-1})$

If V belongs to cyclic code,

$V^{(1)} = (V_{n-1}\ V_0\ V_1\ V_2 \dots V_{n-2})$

$V^{(2)} = (V_{n-2}\ V_{n-1}\ V_0\ V_1 \dots V_{n-3})$

- - - - -

- - - - -

- - - - -

$V^{(i)} = (V_{n-i}\ V_{n-i+1} \dots V_0\ V_1\ V_2 \dots V_{n-i-1})$

Obtained by shifting V cyclically successive and also code vector of C.

$C = DG$
$\downarrow$
$V = DG$
(message M)

$V(x) = V_0 + V_1 x + V_2 x + \dots V_{n-1} x^{n-1}$

Similarly

$V'(x) = V_{n-1} + V_0 x + V_1 x^2 + \dots + V_{n-2} x^{n-1}$

$$V^2(x) = V_{n-2} + V_{n-1}x + V_0 x^2 + \dots + V_{n-3} x^{n-1}$$

$\vdots$

$$V^i(x) = V_{n-i} + V_{n-i+1}x + V_{n-i+2}x^2 + \dots V_{n-i-1} x^{n-1}$$

$$V = 10011 = 1 + x^3 + x^4$$

$$V' = 11001 = 1 + x + x^4$$

Properties of cyclic codes:

1. For a (n, k) cyclic code there exists a generator polynomial of degree $(n-k)$ given by

$$g(x) = g_0 + g_1 x + g_2 x^2 + \dots g_{n-k} x^{n-k}$$

Generator polynomial is unique in that it is the only code vector polynomial of minimum degree $(n-k)$.

2. The generator polynomial $g(x)$ of a (n, k) cyclic code is a factor of $x^n + 1$

$$x^n + 1 = g(x) h(x)$$

$h(x)$ is another polynomial of degree k - parity check polynomial.

3. If $g(x)$ is a polynomial of degree $(n-k)$ and is a factor of x^n+1 then it generates (n, k) cyclic code.

4. The code vector polynomial $V(x)$ can be found by $V(x) = D(x) g(x)$

$D(x)$ - message vector polynomial of degree k

$$d(x) = d_0 + d_1 x + d_2 x^2 + \dots + d_{k-1} x^{k-1}$$

This method generates non systematic cyclic codes.

5. To generate a Systematic cyclic code the remainder polynomial $R(x)$ is obtained from division of $x^{n-k} D(x)$ by $g(x)$. The coefficient of $R(x)$ are placed in the begining of code vector followed by coefficients of message polynomial $D(x)$ to get code vector

$\boxed{\text{n bit code - Vector}}$

$\leftarrow$ Coefficient of $R(x) \rightarrow \leftarrow$ Coefficient of $D(x) \rightarrow$

Generator and Parity Check Matrices of (7,4) cyclic codes

$$g(x) = g_0 + g_1 x + g_2 x^2 + \dots + g_{n-k} x^{n-k}$$

$$\boxed{g(x) = 1 + x + x^3}$$

$$g(x) = 1 + 1(x) + 0(x^2) + 1(x^3) + 0(x^4) + 0(x^5) + 0(x^6)$$
$$= 1101000$$

$x^0\ x^1\ x^2\ x^3\ x^4\ x^5\ x^6$

$$xg(x) = x(1 + x + x^3)$$
$$= x + x^2 + x^4 \Rightarrow 0110100$$

$$x^2 g(x) = x^2(1 + x + x^3)$$
$$= x^2 + x^3 + x^5 \Rightarrow 0011010$$

$$x^3 g(x) = x^3(1 + x + x^3)$$
$$= x^3 + x^4 + x^6$$
$$= 0001101$$

6/3/24

Q Generator matrix

$$G = \begin{array}{c} g(x) \\ xg(x) \\ x^2 g(x) \\ x^3 g(x) \end{array} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$G = [P : I]$

The Generator matrix G is constructed in not a Symmetric form that cannot be visualised in $[P \; I_K]$. ~~Last~~ four elements It can be transformed into Systematic by row transformation.

$$r_3 = r_3 \oplus r_1 \qquad r_4 = r_4 \oplus r_1$$
$$r_4 \oplus r_2$$

$$\begin{array}{r} 1101000 \; \oplus \\ 0011010 \\ \hline 1110010 \end{array}$$

$$\begin{array}{r} 1101000 \; \oplus \\ 0001101 \\ \hline 1100101 \end{array}$$

$$\begin{array}{r} 0110100 \\ \cancel{0001010} \; \oplus \\ 1100101 \\ \hline 1010001 \end{array}$$

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$D = [d_0 \; d_1 \; d_2 \; d_3]$

$V = [D][G]$

$$V = [d_0 \; d_1 \; d_2 \; d_3] \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$V_1 = d_0 \oplus d_2 \oplus d_3$ $\qquad V_4 = d_0$ $\qquad V_7 = d_3$

$V_2 = d_0 \oplus d_1 \oplus d_2$ $\qquad V_5 = d_1$

$V_3 = d_1 \oplus d_2 \oplus d_3$ $\qquad V_6 = d_2$

Messages				Code word						
d_0	d_1	d_2	d_3	V_1	V_2	V_3	V_4	V_5	V_6	V_7
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	0	1	0	0	0	1
0	0	1	0	1	1	1	0	0	1	0
0	0	1	1	0	1	0	0	0	1	1
0	1	0	0	0	1	1	0	1	0	0
0	1	0	1	1	1	0	0	1	0	1
0	1	1	0	1	0	0	0	1	1	0
0	1	1	1	0	0	1	0	1	1	1
1	0	0	0	1	1	0	1	0	0	0
1	0	0	1	0	1	1	1	0	0	1
1	0	1	0	0	0	1	1	0	1	0
1	0	1	1	1	0	0	1	0	1	1
1	1	0	0	1	0	1	1	1	0	0
1	1	0	1	0	0	0	1	1	0	1
1	1	1	0	0	1	0	1	1	1	0
1	1	1	1	1	1	1	1	1	1	1

Systematic code representation: Cyclic code

In the Systematic form, first 3 bits are the redundant or check bits & the last 4 are message bit. The check bits are obtained from the remainder polynomial $R(x)$ where

$$\boxed{R(x) = x^{n-k} D(x) / g(x)}$$

Ex:

Let $D = \overset{0}{1}\overset{1}{0}\overset{2}{0}\overset{3}{1}$

$= 1 + x^3$

$$R(x) = \left(x^3 (1 + x^3)\right) / g(x)$$
$$= (x^3 + x^6) / g(x)$$

$$g(x) = 1+x+x^3$$
$$= x^3+x+1$$

$$\begin{array}{r|l} & x^3+x \\ \hline x^3+x+1 & x^6+x^3 \\ & x^6+x^4+x^3 \\ \hline & x^4 \\ & x^4+x^2+x \\ \hline & x^2+x \end{array}$$

$\therefore$ $R = \underline{011}$

Code word, $V = R \cdot D$
$= \underline{0111001}$

Ex: $D = 1101$
$= 1+x+x^3$

$$x^{n-k} D(x) = x^3(1+x+x^3)$$
$$= x^3+x^4+x^6$$
$$= x^6+x^4+x^3$$

$$R(x) = [x^{n-k} D(x)] / g(x)$$

$$\begin{array}{r|l} & x^3 \\ \hline x^3+x+1 & x^6+x^4+x^3 \\ & x^6+x^4+x^3 \\ \hline & 0 \end{array}$$

$V = R \cdot D$
$= \underline{0001101}$

3/3/24 Parity check matrix [$\cancel{H}$ h]

$\cancel{x^{n}+1} = g(x)\,\cancel{H}(x)$ $\qquad x^n+1 = g(x)h(x)$

for a (7,4) cyclic code, we have

$n = 7$

$\cancel{x^{7+1}} = x^7+1 = g(x)\,h(x)$

the parity check polynomial $h(x)$

$$h(x) = \frac{x^7+1}{g(x)}$$

g(x) not given

$g(x) = 1+x+x^3$

$$
\begin{array}{r|l}
 & x^4+x^2+x+1 \\
\hline
x^3+x+1 & x^7+1 \\
 & x^7+x^5+x^4 \\
\hline
 & x^5+x^4+1 \\
 & x^5+x^3+x^2 \\
\hline
 & x^4+x^3+x^2+1 \\
 & x^4+x^2+x \\
\hline
 & x^3+x+1 \\
 & x^3+x+1 \\
\hline
\end{array}
$$

$h(x) = x^4+x^2+x+1$ ✓

The reciprocal of $h(x)$ is defined as $\cancel{x^k[h(x)]^{-1}}$ $x^4h(x^{-1})$. This polynomial is also a factor of $1+x^n$

Let us consider $x^4[h(x^{-1})]$ for a (7, 4) cyclic code and two cyclic shifted version.

$$h(x) = 1 + x + x^2 + x^4$$

$$h(x^{-1}) = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^4}$$

3 redundant bits; (highest power onwards)

Code $\Rightarrow$ start from constant i.e; $x^0 + x^1 + \dots x$

$x^4 h(x^{-1}) = x^4 + x^3 + x^2 + 1$ $\xleftarrow{1st}$ → 1011100

$x^5 h(x^{-1}) = x^5 + x^4 + x^3 + x$ → 0101110

$x^6 h(x^{-1}) = x^6 + x^5 + x^4 + x^2$ → 0010111

The parity check matrix which is a $(n-k)n$ matrix, (row column)

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

But it is not in the standard form, the standard form equation is, $[I_{n-k}\ P^T]$

$= [I_3\ P^T]$ $\Rightarrow r_1 = r_1 \oplus r_3$

R_1: 1011100 $\oplus$
R_3: 0010111

1001011

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

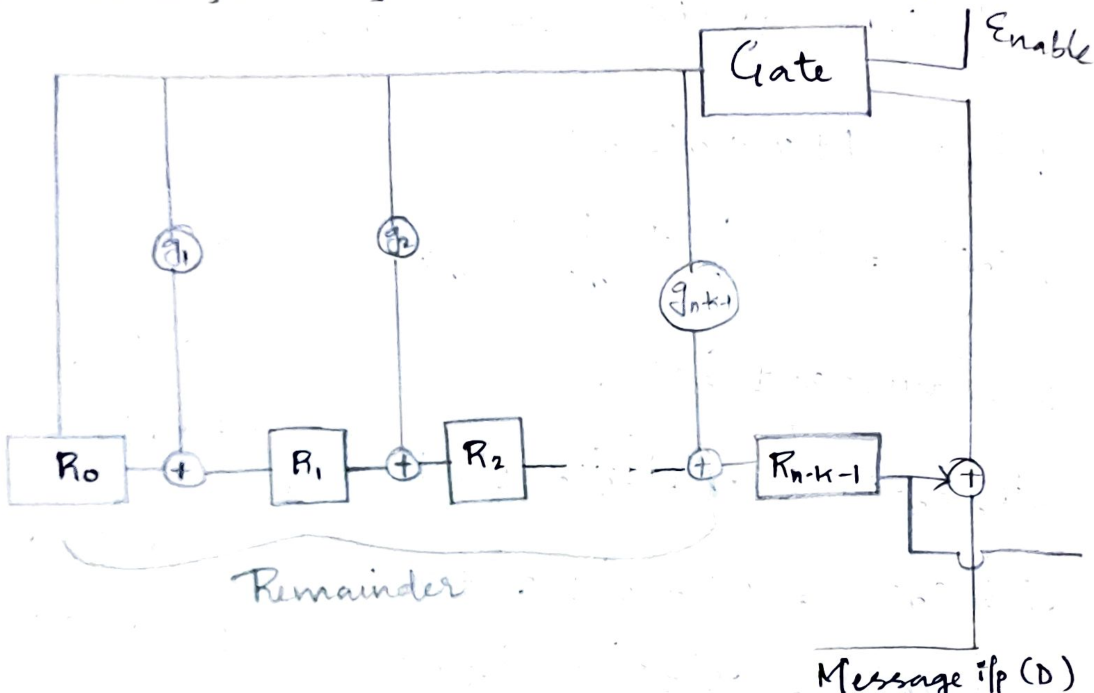
$$P^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$G = [P\ \ I]$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Encoding using n-k bit Shift register.



Q: Design an Encoder for a 7, 4 binary code generated by $g(x) = 1 + x + x^3$ and verify its operation using the message Vectors 1001 and 1011

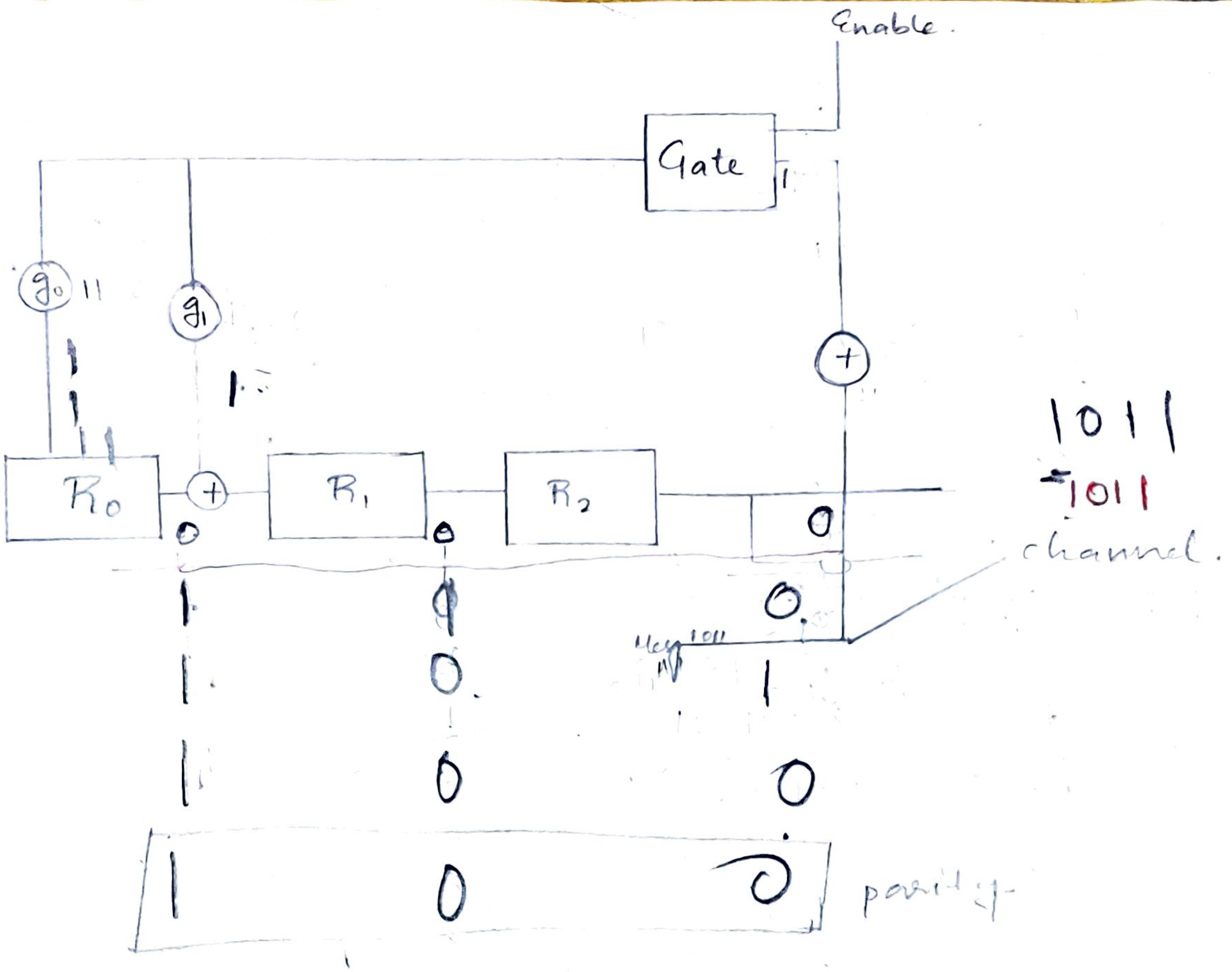
Sol: In general, the generator polynomial equation $g(x)$ is given by,

$$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots + g_{n-k} x^{n-k}$$

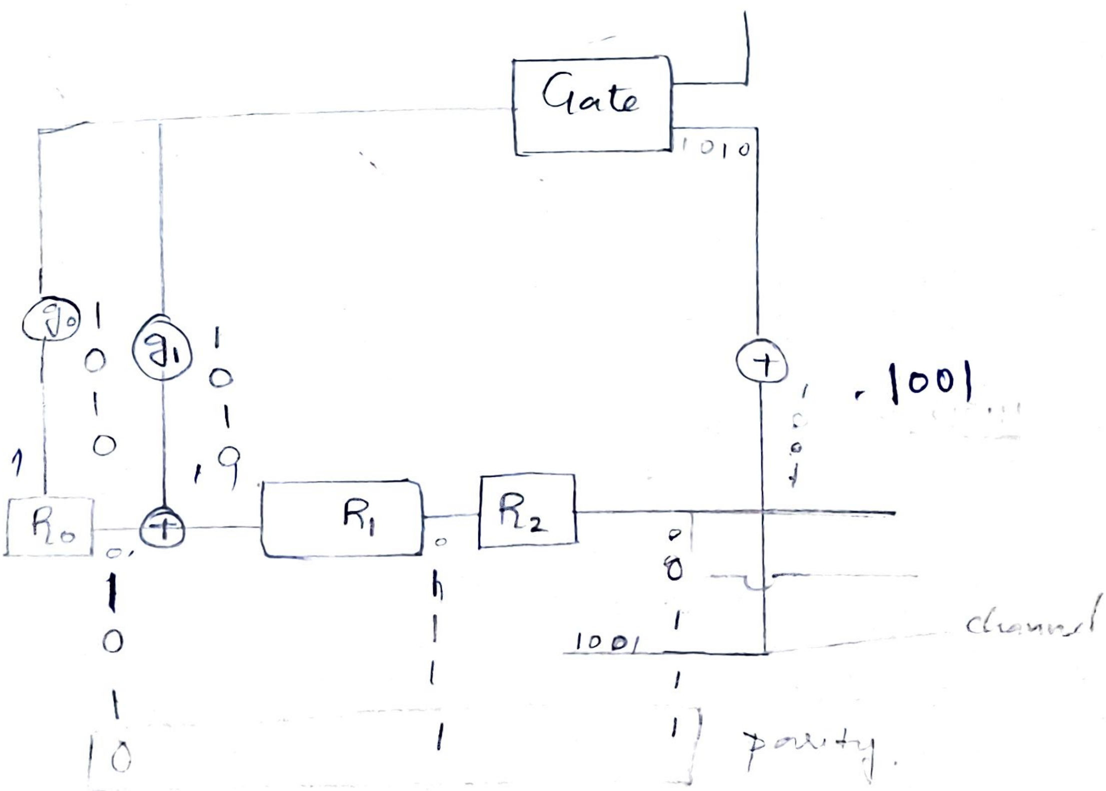
$n = 7$, $k = 4$, $R = 3$ No. of filp flop = 3

$$g(x) = 1 + x + x^3$$

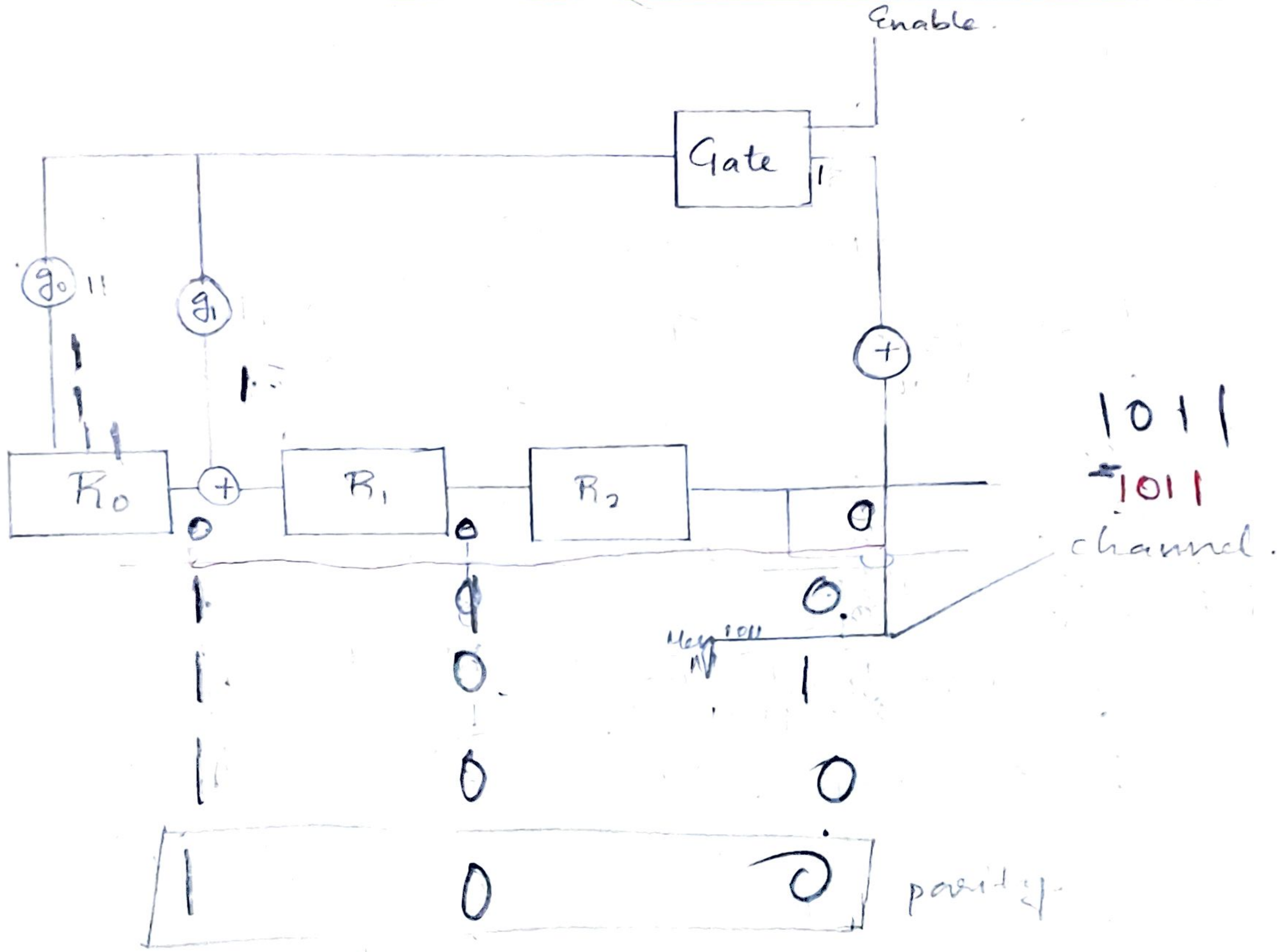
$g_0 = 1$ $g_1 = 1$ $g_2 = 0$ $g_3 = 1$



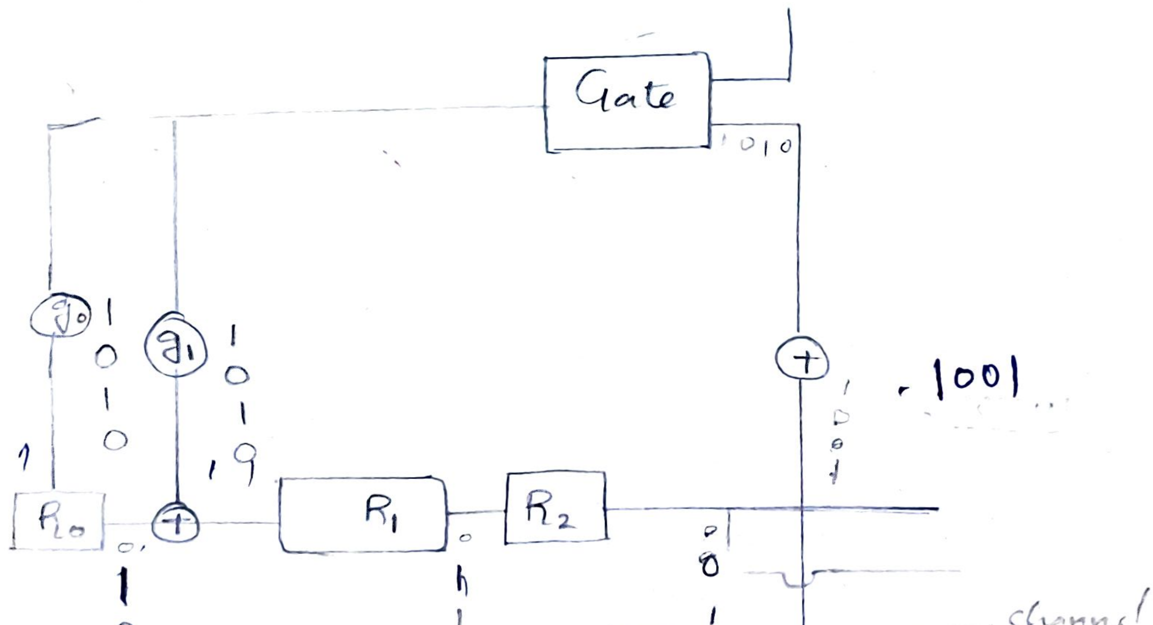
Codeword = 1001011



Code word = 0111001



Codeword = 1001011



Codeword = 0111001

23/3/24

Decoding Circuit

Q. For a (7, 4) cyclic code, the received Vector $r(x)$ is 1110101 and the generated polynomial $g(x) = 1 + x + x^3$. Draw the Syndrome calculator circuit and correct the single error in the received vector.

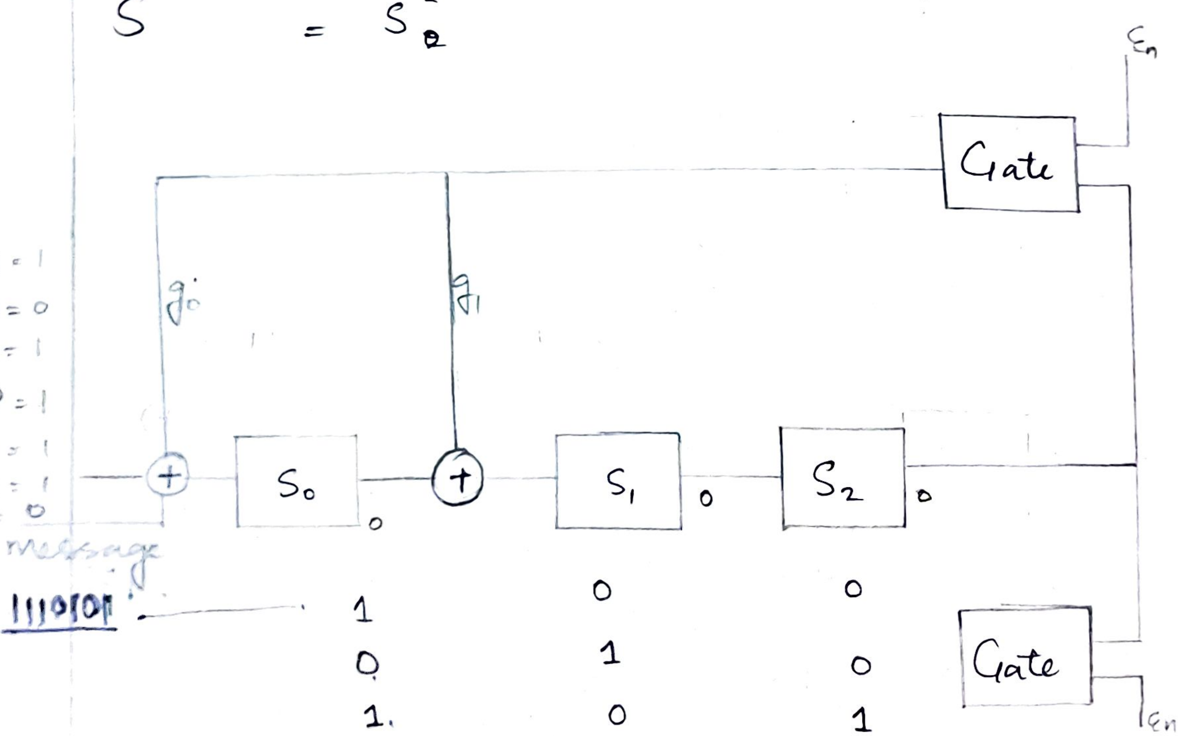
Sol: $g(x) = 1 + x + x^3$

$g_0 = 1 \quad g_1 = 1 \quad g_2 = 0 \quad g_3 = 1$

$(n-k) = (7-4) = \underline{\underline{3}}$

$S^{n-k-1} = S_2^2$

$0 \oplus 1 = 1$
$0 \oplus 0 = 0$
$0 \oplus 1 = 1$
$1 \oplus 0 = 1$
$0 \oplus 1 = 1$
$0 \oplus 1 = 1$
$1 \oplus 1 = 0$



S_0	S_1	S_2
1	0	0
0	1	0
1	0	1
1	0	0
1	1	0
1	1	1
0	0	1

Received = 1110101
         0010000
         1100101

Corrected code word = $\underline{\underline{1100101}}$

6) Generator polynomial of a (15, 7) code is $g(p) = p^8 + p^7 + p^6 + p^4 + 1$. Find the code vector in systematic form for the message vector 1101100.

Sol: $g(p) = p^8 + p^7 + p^6 + p^4 + 1$

$$\begin{array}{ccccccccc} & x^0 & x^1 & x^2 & x^3 & x^4 & x^5 & x^6 \\ D(p) = & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}$$

$= 1 + p + p^3 + p^4$

$p^{n-k} D(p) = p^8 (1 + p + p^3 + p^4)$

$= p^8 + p^9 + p^{11} + p^{12}$

$$\begin{array}{r|l} & p^4 + p^2 + 1 \\ \hline p^8 + p^7 + p^6 + p^4 + 1 & \cancel{p^{12}} + \cancel{p^{11}} + p^9 + \cancel{p^8} \\ & \cancel{p^{12}} + \cancel{p^{11}} + p^{10} + \cancel{p^8} + p^4 \\ \hline & \cancel{p^{10}} + \cancel{p^9} + \cancel{p^8} + p^4 \\ & \cancel{p^{10}} + \cancel{p^9} + p^8 + p^6 + p^2 \\ \hline & \cancel{p^8} + \cancel{p^6} + \cancel{p^4} + p^2 \\ & \cancel{p^8} + p^7 + \cancel{p^6} + \cancel{p^4} + 1 \\ \hline & p^7 + p^2 + 1 \end{array}$$

$$\begin{array}{cccccccccccccccc} & p^0 & p^1 & p^2 & p^3 & p^4 & p^5 & p^6 & p^7 \\ \text{Codeword} = & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}$$

Decoder.

Shift	input	P_0	P_1	P_2	o/p	Remarks
0	1110101	0	0	0	Reset	
1	111010	1	0	0	1	x^9
2	11101	0	1	0	0	x^8
3	1110	1	0	1	1	x^7
4	111	1	0	0	0	x^6
5	11	1	1	0	1	x^5
6	1	1	1	1	1	x^4
7	-	0	0	1	1	x^3
8	-	0	0		1	x^2
9	-	0			0	x^1
10		-			0	x^0

1. Obtain a generator matrix of a (7, 4) cyclic code for the generated polynomial p^3+p^2+1 in non-Systematic and using that find the codeword for the message vector 1100 and

$$\cancel{G(3)} = g(p) = p^3+p^2+1$$

$$1100 = p^3+p^2$$

$$M(p) = p^3+p^2$$

$$x(p) = M(p)\cdot g(p)$$
$$= (p^3+p^2)(p^3+p^2+1)$$
$$= p^6+p^5+p^3+p^5+p^4+p^2$$
$$= p^6+(p^5 \oplus p^5)+p^4+p^3+p^2$$

$$= p^6 + p^4 + p^3 + p^2$$

$$= 1011100$$

$$G(3) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$M(p) = 1111 = p^3 + p^2 + p + 1$$

$$x(p) = M(p)\, g(p)$$

$$= (p^3 + p^2 + p + 1)(p^3 + p^2 + 1)$$

$$= p^6 + p^5 + p^4 + p^3 + p^5 + p^4 + p^3 + p^2 + p^3 + p^2$$
$$+ p + 1$$

$$= p^6 + (1 \oplus 1) p^5 + (1 \oplus 1) p^4 + (1 \oplus 1 \oplus 1) p^3$$
$$+ (1 \oplus 1) p^2 + p + 1$$

$$= p^6 + p^3 + p + 1$$

$$= 1001011$$

$$G(3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

α powers	Binary representation (4 bit) α^3	α^2	α^1	α^0
α^{15} α^0	0	0	0	1
α^{16} α^1	0	0	1	0
α^{17} α^2	0	1	0	0
α^{18} α^3	1	0	0	0
α^{19} α^4	0	0	1	1
α^{20} α^5	0	1	1	0
α^{21} α^6	1	1	0	0
α^{22} α^7	1	0	1	1
α^{23} α^8	0	1	0	1
α^{24} α^9	1	0	1	0
α^{25} α^{10}	0	1	1	1
α^{26} α^{11}	1	1	1	0
α^{27} α^{12}	1	1	1	1
α^{28} α^{13}	1	1	0	1
α^{29} α^{14}	1	0	0	1

In Single error correcting code $(t=1)$

The generator polynomial $g(x) = \mathrm{LCM}[\underline{M_1(x)}, M_3(x) \ldots M_{2t-1}(x)]$

When $t=1$

$g(x) = \mathrm{LCM}(M_1(x))$

To find $M_1(x)$ the minimum polynomial

$M_1(x) = M_1(\alpha^i)$

$x^4 = (\alpha^1)^4 \qquad x^2 = (\alpha^1)^2$

$x^3 = (\alpha^1)^3 \qquad x = (\alpha^1)^1$

$1 = 1$

$x^4 = (\alpha)^4 = 0011$ $\quad (\alpha^2) = 0100$ $\quad (\alpha^0) = 0001$

$(\alpha)^3 = 1000$ $\quad (\alpha^1) = 0010$

0011
1000
0100
0010
0001
~~1100~~

$M_1(x) = x^4 + x + 1$

$\therefore g(x) = x^4 + x + 1$

$\cancel{n-k}$ $n-K$ is the ~~ge~~ degree of $g(x) = 4$

$n - K = 4$ $\quad$ 0011

$15 - K = 4$ $\quad \therefore K = 11$

The single error correcting BCH code $(15, 11)$ cyclic code with $g(x) = x^4 + x + 1$ and the

$\boxed{d_{min} = 2t + 1}$ $\quad \therefore d_{min} = 3$

d_{min} is the weight of $g(x)$.

For a double error correcting code $(t = 2)$

$$g(x) = \text{LCM}\,[M_1(x), M_3(x) \ldots M_{2t-1}(x)]$$

$$= \text{LCM}\,[M_1(x)\ M_3(x)]$$

x^4 0011
x^3 0100
x^2 0001
x 1000
1 0010

$M_1(x) = M_1(\alpha^1)$ $\quad M_3(x) = M_3(\alpha^3)$

$M_1(\alpha^1)$

$x^4 = (\alpha^1)^4 = 0011$

$x^3 = (\alpha^1)^3 = 1000$

$x^2 = (\alpha^1)^2 = 0100$

$x^1 = \alpha^1 = 0010$

$1 = \alpha^0 = 0001$

$M_3(\alpha^3)$

$x^4 = (\alpha^3)^4 =$ ~~1010~~ 1111

$x^3 = (\alpha^3)^3 = 1010$

$x^2 = (\alpha^3)^2 = 1100$

$x^1 = (\alpha^3)^1 = 1000$

$x^0 = \alpha^0 = 0001$

$M_1(x) = x^4 + x + 1$

$M_3(x) = x^4 + x^3 + x^2 + x + 1$

1	1	1	1
1	0	1	0
1	1	0	0
1	0	0	0
0	0	0	1

$g(x) \overset{LCM}{=} (M_1(x) \cdot M_3(x))$

$= x^8 + x^7 + x^6 + \cancel{x^5} + \cancel{x^4} + \cancel{x^5} + \cancel{x^4} + \cancel{x^3} + \cancel{x^2} + \cancel{x} +$

$x^4 + \cancel{x^3} + \cancel{x^2} + \cancel{x} + 1$

$= \underline{x^8 + x^7 + x^6 + x^4 + 1}$

$n - k = 8$

$\underline{K = 7}$

The double error correcting BCH code (15, 7) cyclic code with $g(x) = x^8 + x^7 + x^6 + x^4 + 1$ and $d_{min} =$

For a triple error correcting code $(t = 3)$

$g(x) = LCM(M_1(x), M_3(x) \ldots M_{2t-1}(x))$

$t = 3 \Rightarrow g(x) = LCM(M_1(x), M_3(x), M_5(x))$

$M_5(x) = M_5(\alpha^5)$

$\underline{M_5(\alpha^5)}$

$x^4 = (\alpha^5)^4 = 0110$

$x^3 = (\alpha^5)^3 = 0001$

$x^2 = (\alpha^5)^2 = 0111$

$x = (\alpha^5)^1 = 0110$

$x^0 = \alpha^0 = 0001$

$x^4 + x^3 + x^2 + 1$

0	1	1	0
0	0	0	1
0	1	1	1
0	1	1	0
0	0	0	1

$M_5(x) = x^2 + x + 1$

$g(x) = (M_1(x)\, M_3(x)\, M_5(x))$

$= (x^4 + x + 1)(x^4 + x^3 + x^2 + x + 1)(x^2 + x + 1)$

$= (x^8 + x^7 + x^6 + x^4 + 1)(x^2 + x + 1)$

$= x^{10} + \cancel{x^7} + \cancel{x^8} + x^9 + x^8 + \cancel{x^7} + x^8 + x^7 + x^6 +$

$+ x^6 + x^5 + x^4 + x^2 + x + 1$

$= \underline{x^{10} + x^8 + x^5 + x^4 + x + 1}$

$n-k = 10$ The triple error correcting BCH code

$k = \underline{5}$ (15, 5) cyclic code with $g(x) = x^{10} + x^8 + x^5 + x^4 + x + 1$ and $d_{min} = \underline{7}$

1/4/24 BCH Codes.

* BCH code is a code with best error correcting capability code.
* It is a class of cyclic codes.
* Major advantage of BCH code is the flexibility in the choice of block length n and code rate k/n.

Description:

For any +ve integer $m \geq 3$ there exist a t error correcting B.C.H code with the following parameters:

1. Block length
$$n = 2^m - 1$$
2. No. of message bits, $k \geq n - mt$
3. Minimum Distance, $d_{min} \geq 2t + 1$

If $m \leq 10$, the code is known as Premit BCH Codes.

Encoding of BCH Code.

t; No. of errors to be corrected

n: block length will be given.

Step 1: Select an integer m Such that

$$n = 2^m - 1$$

Step 2: Select an irreducable polynomial of Order 'm'

Step 3: Calculate the generator polynomial $g(x)$ Such that,

$$g(x) = LCM\{M_1(x), M_3(x), M_5(x) \dots M_{(2t-1)}(x)\}$$

where $M_1(x), M_3(x) \dots$ are minimum polynomials based on the irreducable polynomial.

Step 4: find the no. of message bits such that

$$n-k = \text{degree of } g(x).$$

Step 5: Encoding is Same as cyclic code multiply the message polynomial $M(x)$ with x^{n-k}.

Step 6: find the reminder of the Division

$$b(x) = [x^{n-t} m(x)] \% g(x)$$

Step 7: Represent the codeword polynomial as

$$V(x) = x^{n-k} m(x) + b(x)$$

Decoding is optional (Syndrome Decoding)

Step 1: Calculate the Syndrome as

$$b_i(x) = r(x) \% m_i(x)$$

Step 2: Generate the error location polynomial

$$\sigma(x) = (1+\beta_1 x)(1+\beta_2 x) \dots$$

Step 3: Find the roots of the $\sigma(x)$ to get the reciprocal of error location number.

Step 4: Correct the error by complimenting the corresponding bits

REED SOLOMON CODES:

Sub class of BCH codes, here the encoder operates on a block of data. The parameters are block length $n = 2^m - 1$

Message Size $K = n - 2t$

Parity check Size $r = n - k = 2t$ Symbols.

$d_{min} = 2t + 1$ Symbols.

Applications:

QR code, Bar code, Storage like CD, DVD etc.

* Encoding:

Let the message polynomial be $p(x)$ and generator polynomial by $(x - \alpha)(x - \alpha^2) \ldots (x - \alpha^{2t})$, then the code polynomial $c(x) = p(x) \cdot g(x)$

• Error location:

$r(x) \quad d(x)$

$e(x) = d(x) - r(x)$

3/4/24

Module - 5.

Convolutional code.

The convolutional code depends ~~on~~ not only the present message bit but also the previous message bit and that are stored in the shift register

* The convolution codes are codes with memory and we require sequential ckt

* To design the convolutional encoder, the dimension of convolutional encoder is (n, k, m) where n is the no. of o/ps, k is the no. of i/ps, m is the no. of memory elements.

* The rate of convolutional code $R = \frac{k}{n}$

Constraint length.

It is the total no. of message bit and its previous bits which are responsible for the o/p of convolutional encoder. It can be calculated as

$$L = k(m+1)$$

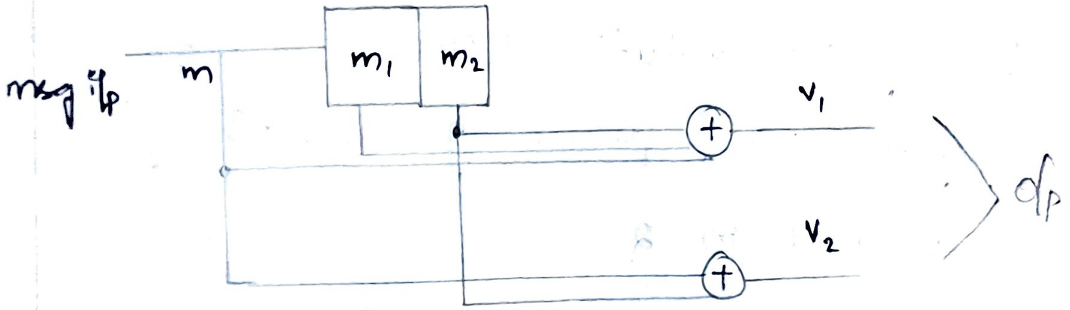
Usually the constraint length will be equal to the no. of elements in the generator sequence.

A (2, 1, 2) convolutional encoder with a generated sequence, $g^{(1)} = \{1, 1, 1\}$, $g^{(2)} = \{1, 0, 1\}$

Sol: $n = 2$, $k = 1$, $m = 2$

No. of i/p = 1   No. of o/p = 2

No. of memory element = 2



$V_1 = U * g^{(1)}$ $V_2 = U * g^{(2)}$

o/p $y = V_1 V_2 \; V_1 V_2 \ldots$

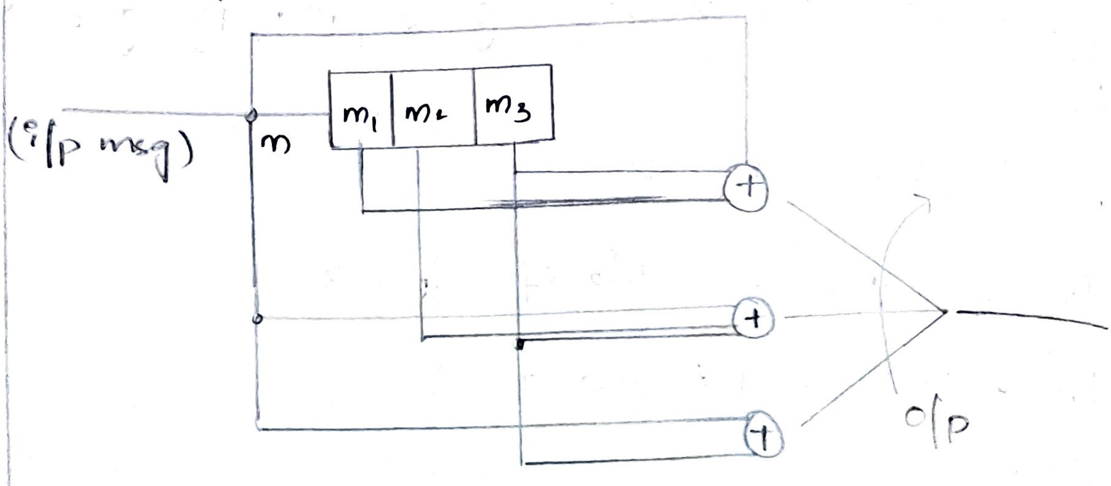
The o/p of the convolutional encoder is calculated as the linear convolution of the i/p sequence and the generator sequence.

Q Design a convolutional encoder of the dimension {3, 1, 3} with a generator sequence $g_1^{(1)}(x) = \{1 + x + x^3\}$ $g_2(x) = 1 + x^2 + x^3$ $g_3(x) = 1 + x^3$

Sol: Given

$n = 3$, $k = 1$, $m = 3$

$g^{(1)} = \{1101\}$, $g^{(2)} = \{1011\}$, $g^{(3)} = \{1001\}$

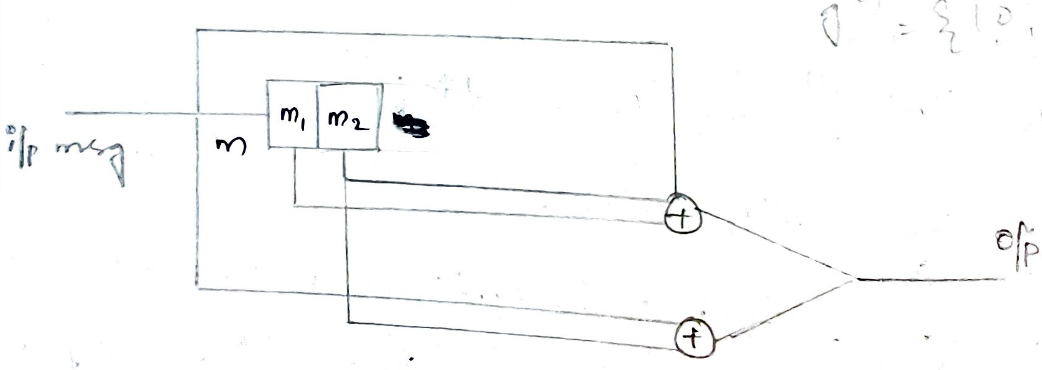


Q Design a convolutional encoder {2, 1, 2} with a generated Sequence $g^{(1)} = \{1\ 1\ 1\}$ $g^{(2)} = \{1\ 0\ 1\}$

$n = 2$ $k = 1$ $m = 2$

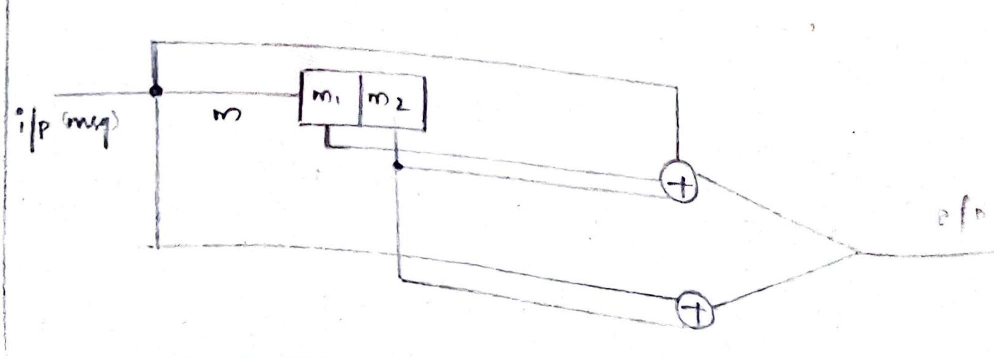
$g^{(1)} = \{1, 1, 1\}$

$g^{(2)} = \{1, 0, 1\}$



Q Draw the convolutional encoder with dimens (2, 1, 2} with a generated Sequence $g^{(1)} = \{1, 1, 1\}$ $g^{(2)} = \{1, 0, 1\}$. Also find the o/p Sequence is U = 100110.

$n = 2,$ $k = 1,$ $m = 2$



$U = \{1\,0\,0\,1\,1\,0\}$

$V_1 = U * g^{(1)}$

$= 100110 * 111$

$$\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}$$

$V_1 = 11110010$

$V_2 = U * g^{(2)}$

$= 100110 * 101$

$$\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}$$

$V_2 = 10111110$

$\therefore$ O/p $y = V_1 V_2 = 11110010\ 10111110$

$y = 11, 10, 11, 11, 01, 01, 11, 00$

5/4/24

State diagram and Trellis diagram of a convolutional Encoder.

Q Draw the State diagram of a convolutional encoder with rate $= 1/3$ and constraint length $= 3$ for the generated Sequence $g^{(1)} = \{1\,0\,0\}$ $g^{(2)} = \{1\,0\,1\}$ $g^{(3)} = \{1\,1\,1\}$

Sol: $R = \frac{k}{n} = \frac{1}{3}$ $k = 1$, $n = 3$

$m = n - k = 2$

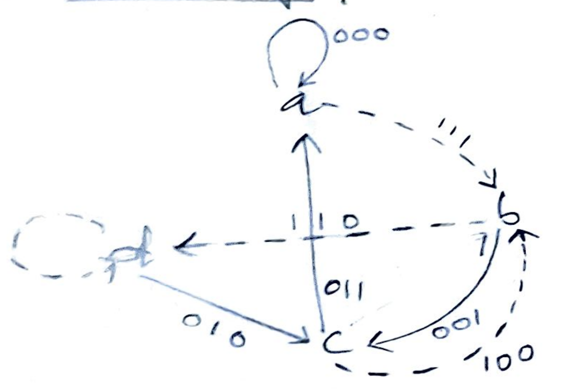
$(n, k, m) = (3, 1, 2)$

$\boxed{\text{No. of State} = 2^m}$

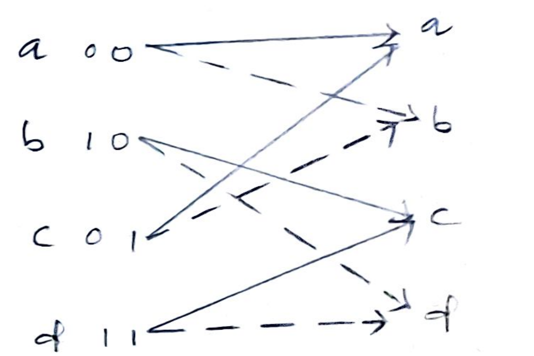
$= 2^2 = 4$

	Current State m_1	m_2	i/p m	o/p V_1	V_2	V_3	Next State m	m_1	State
A	0	0	0	0	0	0	0	0	a
			1	1	1	1	1	0	b
b	1	0	0	0	0	1	0	1	c
			1	1	1	0	1	1	d
c	0	1	0	0	1	1	0	0	a
			1	1	0	0	1	0	b
d	1	1	0	0	1	0	0	1	c
			1	1	0	1	1	1	d

State diagram:



Trellis diagram.

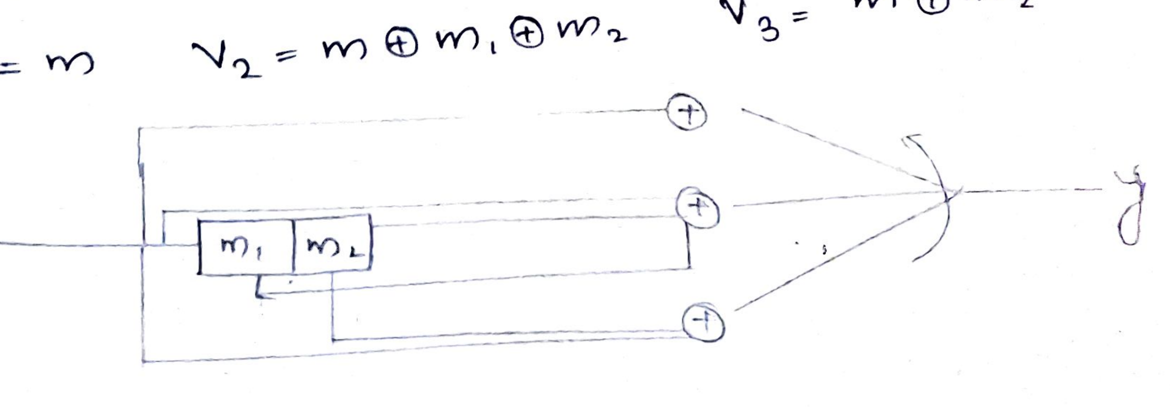


Q) A rate 1/3 convolution encoder has a generated Sequence $g^{(1)} = \{1, 0, 0\}$ $g^{(2)} = \{1, 1, 1\}$ $g^{(3)} = \{1 0 1\}$ Sketch the encoder configuration. Draw the State diagram and Trellis diagram.

Sol: $R = \frac{k}{n} = 1/3$ $\qquad m = n - k = \underline{\underline{2}}$

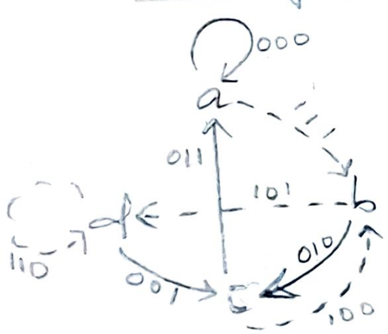
$k = 1 \qquad n = 3 \qquad (n, k, m) = (3, 1, 2)$

$V_1 = m \qquad V_2 = m \oplus m_1 \oplus m_2 \qquad V_3 = m \oplus m_2$

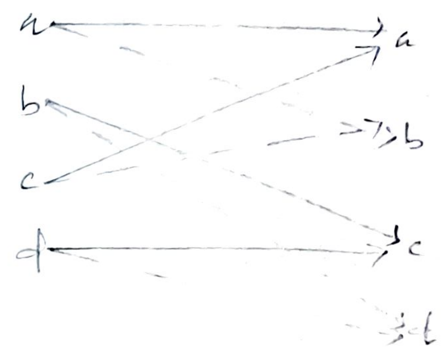


	Current state m_1	Current state m_2	i/p m	o/p V_1	o/p V_2	o/p V_3	Next stage m_1	Next stage m_2	State
a	0	0	0	0	0	0	0	0	a
			1	1	1	1	1	0	b
b	1	0	0	0	1	0	0	1	c
			1	1	0	1	1	1	d
c	0	1	0	0	1	1	0	0	a
			1	1	0	0	1	0	b
d	1	1	0	0	0	1	0	1	c
			1	1	1	0	1	1	d

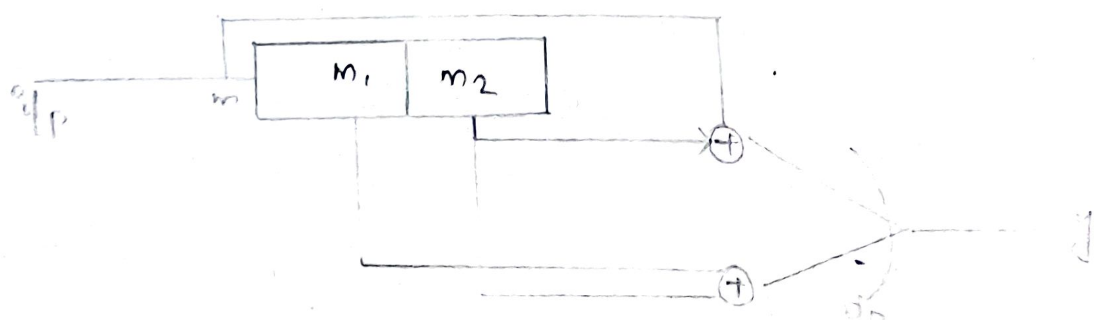
State diagram



Trellis diagram



Q2 For a convolutional encoder with generated Sequence $g^{(1)} = \{1\ 0\ 1\}$ $g^{(2)} = \{0\ 1\ 1\}$.
Draw the encoder circuit



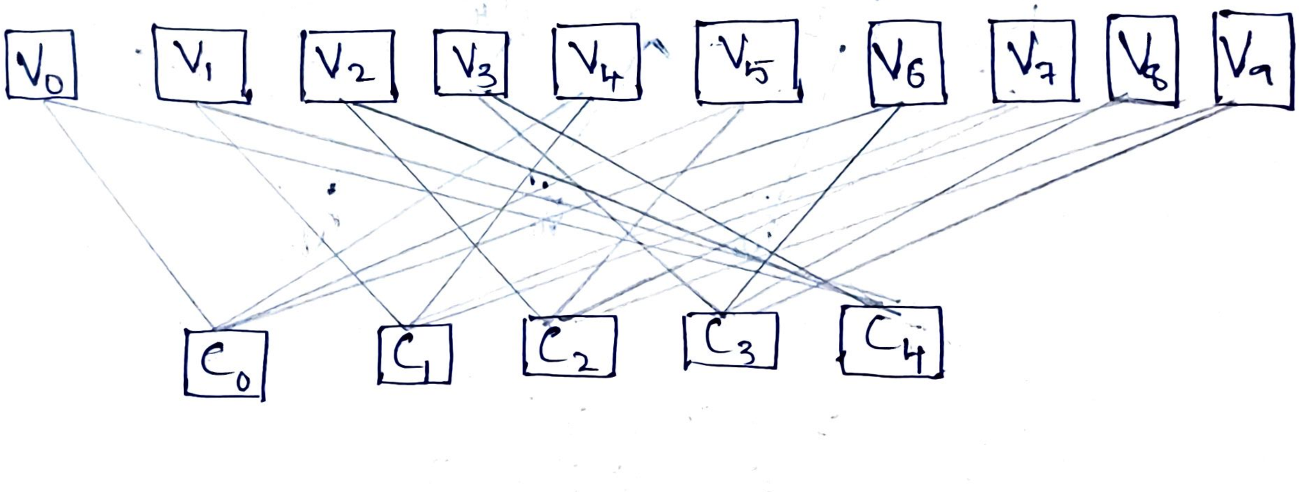
Q In (6, 4) LDPC code $H = \begin{bmatrix} 1&0&0&0&1&1&1&0&0&0 \\ 0&1&0&0&1&0&0&1&1&0 \\ 0&0&1&0&0&1&0&1&0&1 \\ 0&0&0&1&0&0&1&0&1&1 \\ 1&1&1&1&0&0&0&0&0&0 \end{bmatrix}$

Draw the tanner graph.

Sol. $H = \begin{bmatrix} 1&0&0&0&1&1&1&0&0&0 \\ 0&1&0&0&1&0&0&1&1&0 \\ 0&0&1&0&0&1&0&1&0&1 \\ 0&0&0&1&0&0&1&0&1&1 \\ 1&1&1&1&0&0&0&0&0&0 \end{bmatrix}$

No. of message nodes = 10

No. of check nodes = 5



$C_0 = V_0 + V_4 + V_5 + V_6$

$C_1 = V_1 + V_4 + V_7 + V_8$

$C_2 = V_2 + V_5 + V_7 + V_9$

$C_3 = V_0 + V_1 + V_2 + V_3$

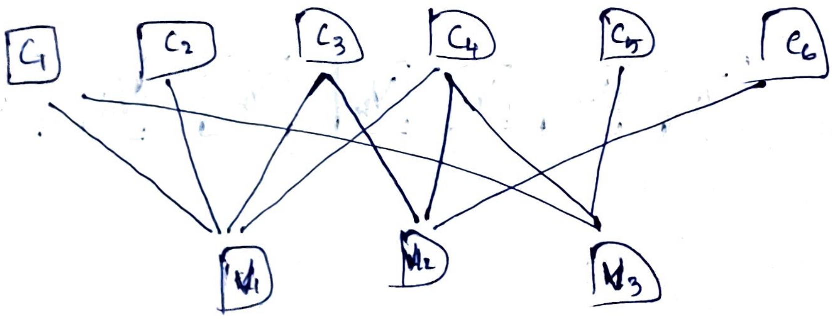
Q Consider (6, 2, 4) LDPC code with the following Parity equation

V_1 $\quad c_1 \oplus c_2 \oplus c_3 \oplus c_4 = 0$

V_2 $\quad c_3 \oplus c_4 \oplus c_6 = 0$

V_3 $\quad c_1 \oplus c_4 \oplus c_5 = 0$

The transmitted msg 101011 through a binary erasure channel is received as _ 0 1 _ 1 1



$$\begin{array}{ccccccc} c_3 & & c_4 & & c_6 & & \\ 1 & \oplus & \_ & \oplus & 1 & = & 0 \end{array}$$

$\underline{\underline{c_4 = 0}}$

$c_1 \oplus c_2 \oplus c_3 \oplus c_4 = 0$

$\underline{1} \oplus 0 \oplus 1 \oplus 0 = 0 \qquad \therefore c_1 = \underline{\underline{1}}$

$\therefore$ Received code word = $\underline{\underline{101011}}$

No error while transmitting.

6) For a (2, 1, 2) convolutional encoder with a generated sequence $g^{(1)} = \{1, 1, 1\}$ and $g^{(2)} = \{1, 0, 1\}$. Draw the trelli's perform the decoding on the trelli's for the received sequence 01101 011010111. Obtain the transmited sequence.

$V_1 = m \oplus m_1 \oplus m_2$ $\quad$ $V_2 = m \oplus m_2$ $\quad$ No. of state $= 2^m = 2^2$
$= \underline{\underline{4}}$

Current State		i/p	o/p		Next Stage		State
m_1	m_2	m	V_1	V_2	m	m_1	
0	0	0	0	0			
		1	1	1			
0	1	0	1	1			
		1	0	0			
1	0	0	0	0			
		1	1	1			
1	1	0					
		1					

Tanner Graph.

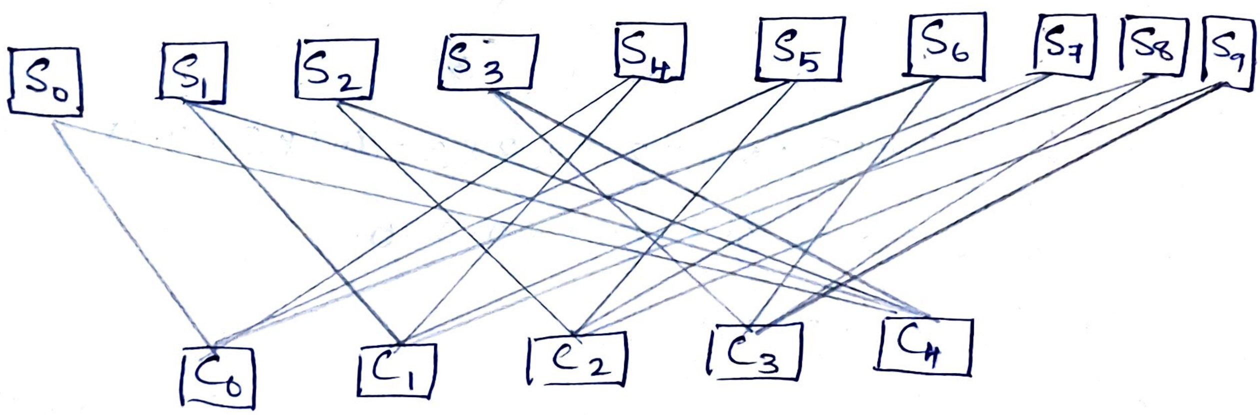
Graphical representation of LDPC codes.

*

* The nodes/msg bits are connected to the check node.

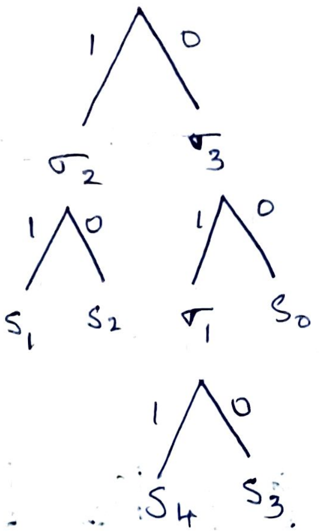
Ex: for a (10, 7, 4) the LDPC code can be given as

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



S_0	0.4	0.4	$\sigma_2 = 0.4$	$\sigma_3 = 0.6$
S_1	0.2	$\sigma_1 = 0.2$	0.4	$\sigma_2 = 0.4$
S_2	0.2	0.2	$\sigma_1 = 0.2$ σ_3	
S_3	0.1	0.2 σ_2		
S_4	0.1 σ_1			

$S_0 = 00 = 2$
$S_1 = 11 = 2$
$S_2 = 10 = 2$
$S_3 = 010 = 3$
$S_4 = 011 = 3$



$$\bar{L} = 0.4\times2 + 0.2\times2 + 0.2\times2 + 0.1\times3 + 0.1\times3$$

$$= 0.8 + 0.4 + 0.4 + 0.3 + 0.3$$

$$= \underline{\underline{2.2}}$$

$$H(s) = 0.4\log_2\frac{1}{0.4} + 0.2\log_2\frac{1}{0.2} + 0.2\log_2\frac{1}{0.2} + 0.1\log_2\frac{1}{0.1} + 0.1\log_2\frac{1}{0.1}$$

$$= 0.5287 + 0.46438 + 0.46438 + 0.332192 + 0.332192$$

$$= 1.457467 + 0.664385$$

$$= \underline{\underline{2.121845}}$$

$$\eta = \frac{H(S)}{\bar{L}} = 0.96447 = \underline{96.4\%}$$

$$\gamma = 1 - \eta = \underline{\underline{0.0355}}$$

(n, k) $k = 4$ $d_{min} = 3$

$$\boxed{n \le 2^{r} - 1}$$ $r = n - k.$

$$n \le 2^{n-k} - 1$$

Trial & Error method.

$n = 4$, $4 \le 2^{0} - 1$
$1 - 1$ X

$6 \le 2^{4} - 1$ X

$7 \le 2^{3} - 1 = 8 - 1 = 7.$

$\therefore \underline{\underline{n = 7.}}$

$d_{min} = 3$

3 bit msg

0 0 0 X

~~0 0 1~~

~~0 1 0~~

0 1 1 ✓

~~1 0 0~~

1 0 1 ✓

1 1 0 ✓

1 1 1 ✓

$$H^{T} = \left[\frac{P}{I}\right]$$

$$= \begin{bmatrix} P \\ 1\;0\;0 \\ 0\;1\;0 \\ 0\;0\;1 \end{bmatrix}$$

$$H^{T} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The parity check matrix of a 7, 4 LBC is given as

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Construct the code word Show that this is a hamming code.

$$H = [P^T \; I]$$

$$P^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \therefore P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[C] = [D][G] = [d_1 \; d_2 \; d_3 \; d_4] \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \{d_1, d_2, d_3, d_4, d_1 \oplus d_3 \oplus d_4, d_1 \oplus d_2 \oplus d_4, d_2 \oplus d_3 \oplus$$

C_1
C_2
C_3
C_4
C_5
C_6
C_7
C_8
C_9
C_9
C_{10}
C_{11}
C_{12}
C_{13}
C_{14}
~~C_{15}~~
C_{15}
C_{16}

$d_1\, d_2\, d_3\, d_4$
0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1

0 0 0 0	0	0	0
0 0 0 1	1	1	1
0 0 1 0	1	0	1
0 0 1 1	0	1	0
0 1 0 0	1	1	1
0 1 0 1	[illegible]	0	0
0 1 1 0	1	1	0
0 1 1 1	0	0	1
1 0 0 0	1	1	0
1 0 0 1	0	0	1
1 0 1 0	0	[illegible]	1
1 0 1 1	1	0	0
1 1 0 0	1	0	1
1 1 0 1	0	1	0
1 1 1 0	0	0	0
1 1 1 1	1	1	1

$r = n - k$
$= 7 - 4$
$= \underline{3}$

$n \leq 2^{r} - 1$
$7 \leq 2^{3} - 1$
$= 7$

$\therefore$ (7, 4) is a Hamming code.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$c_4 = d_1 \oplus d_3$$

$$c_5 = d_2 \oplus d_3$$

$$c_6 = d_1 \oplus d_2$$

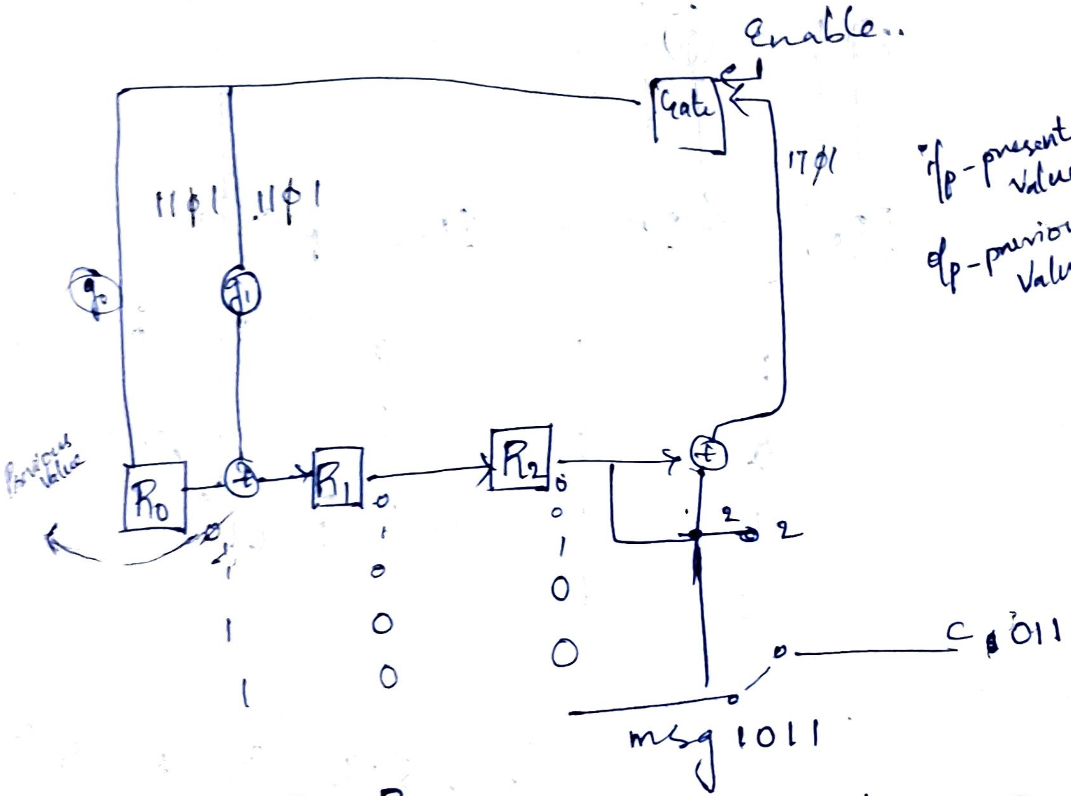
Msg	Codeword
0 0 0	0 0 0
0 0 1	1 1 0
0 1 0	0 1 1
0 1 1	1 0 1
1 0 0	1 0 1
1 0 1	0 1 1
1 1 0	1 1 0
1 1 1	0 0 0

(7, 4) $g(x) = 1 + x + x^3$ msg (1001)
(1011)

$g(x) = g_0 + g_1 x + g_2 x^2 + \dots + g_{n-k} x^{n-k}$

Given, $g(x) = 1 + x + x^3$

$\cdot\ g_0 = 1 \quad g_1 = 1 \quad g_2 = 0 \quad g_3 = 1$

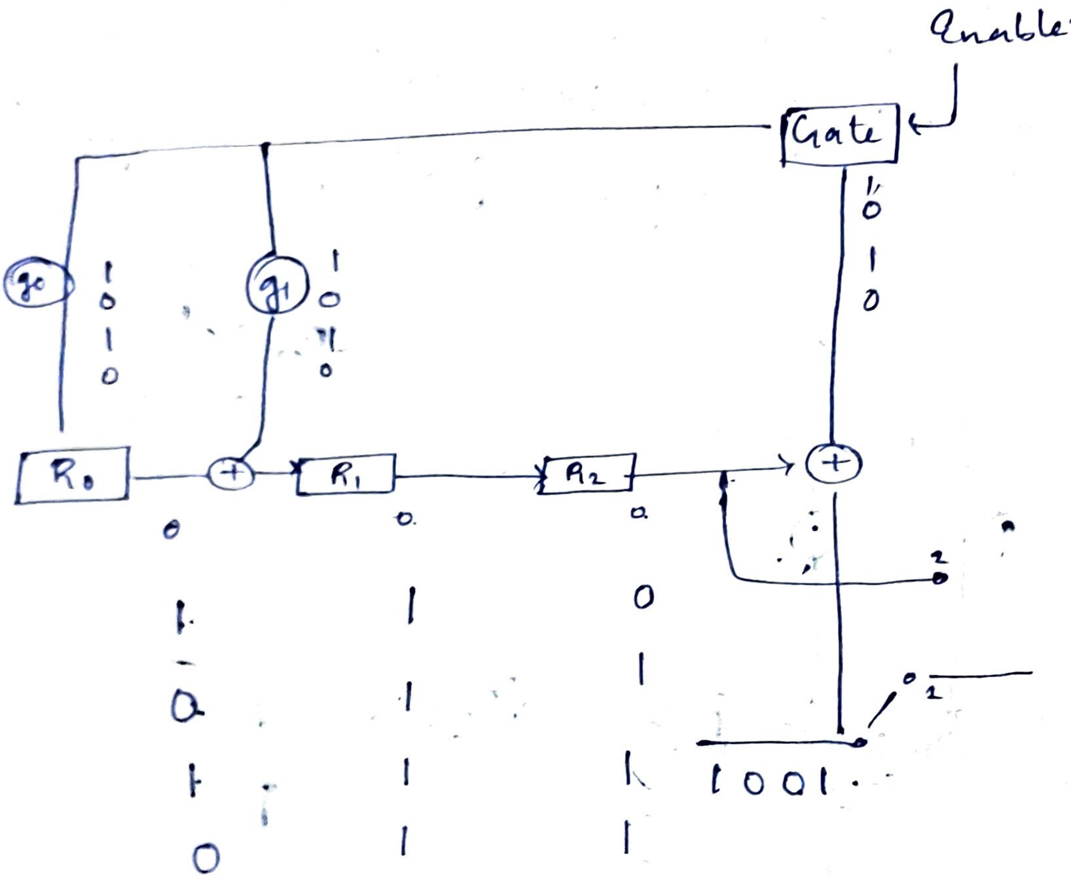


Ip - present value
Op - previous value

	R_0	R_1	R_2
	0	0	0
1 →	1	(1⊕0) 1	0
1 →	1	0	1
0 →	1	0	0
1 →	1	0	0

No. of shifts	Ip	R_0 R_1 R_2	Remainder
	0	0 0 0	
	1	1 1 0	
1	1	1 0 1	
2	0	1 0 0	
3	1	1 0 0	
4		1 0	0
5	x	1	0
6	x		
7	x		1

Codeword = 1001011



$1 \oplus 0 = 1$
$0 \oplus 0 = 0$
$0 \oplus 1 = 1$
$1 \oplus 1 = 0$

	R_0	R_1	R_2
	0	0	0
1 →	1	1	0
0 →	0	1	1
0 →	1	1	1
1 →	0	1	1

No. of Shifts	i/p	R_0	R_1	R_2	Remainder
		0	0	0	
1	1	1	1	0	
2	0	0	1	1	
3	0	1	1	1	
4	1	0	1	1	
			0	1	1
5	x			0	1
6	x				0
7	x				

Codeword = 0111001

Decoder:

$(7,4)$ (ce) $\quad g(x) = 1 + x + x^3 \qquad h(x) = 1110101$

$\boxed{n-k=3}$

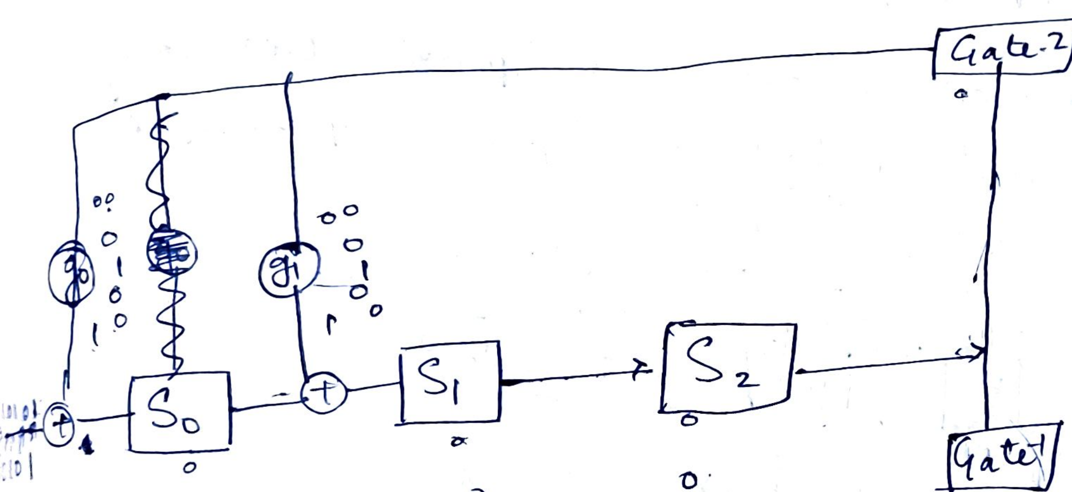
$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3$

given $g(x) = 1 + x + x^3 \quad \therefore g_0 = 1 \quad g_1 = 1$

$g_2 = 0 \quad g_3 = 1$

$\boxed{S_0 — S_{n-k-1}}$

enable.



	S_0	S_1	S_2
1+0	1	0	0
0+0	0	1	0
	1	0	1
	1	0	0
	1	1	0
	1	1	1
	0	0	1

	S_0	S_1	S_2
	0	0	0
1→	1	0	0
0→	0	1	0
1→	1	0	1
0→	1	0	0
1→	1	1	0
1→	1	1	1
1→	0	0	1

No-of shifts	i/p	shift S_0	S_1	S_2	Comments
		0	0	0	
1	1	1	0	0	
2	0	0	1	0	
3	1	1	0	1	
4	0	1	0	0	
5	1	1	1	0	
6	1	1	1	1	
7	1	0	0	1	← Indicates error
8	0	1	1	0	
9	0	[illegible]	[illegible]	[illegible]	
10	0				
11	0				
12	0				

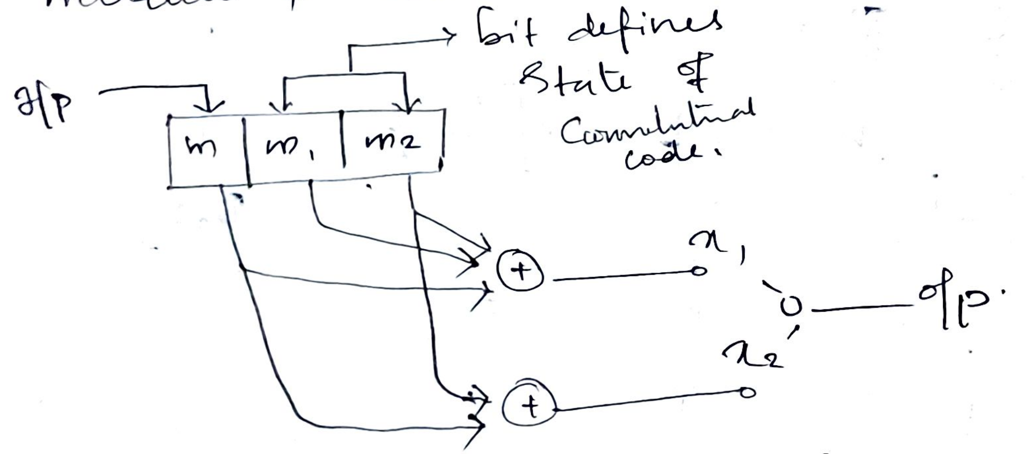
1110101 +
0010000

1100101

Convolutional codes.

(n, k, m)

⇒ depends on previous i/p also (m) along with k.
⇒ redundant bits are generated ~~used in~~ using modulo 2 addition.



$x_1 = m \oplus m_1 \oplus m_2$ $\quad$ $x_2 = m \oplus m_2$

——— i/p 0

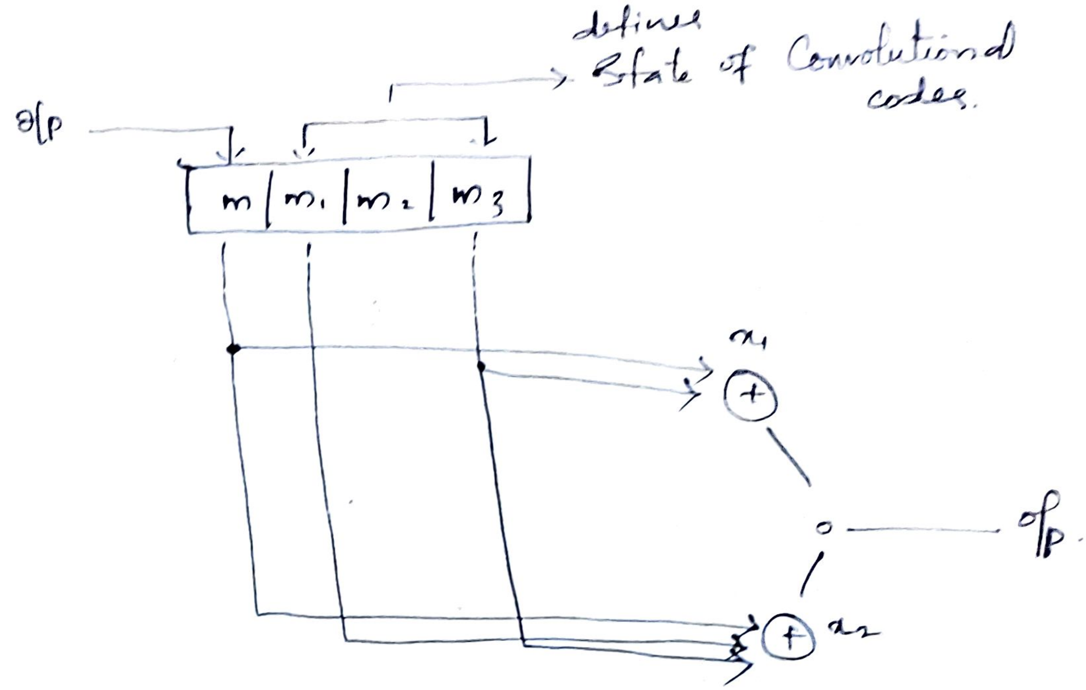
- - - - i/p 1

$(2, 1, 3)$

n, k, m

$g^{(1)} = (1\ 0\ 0\ 1)$

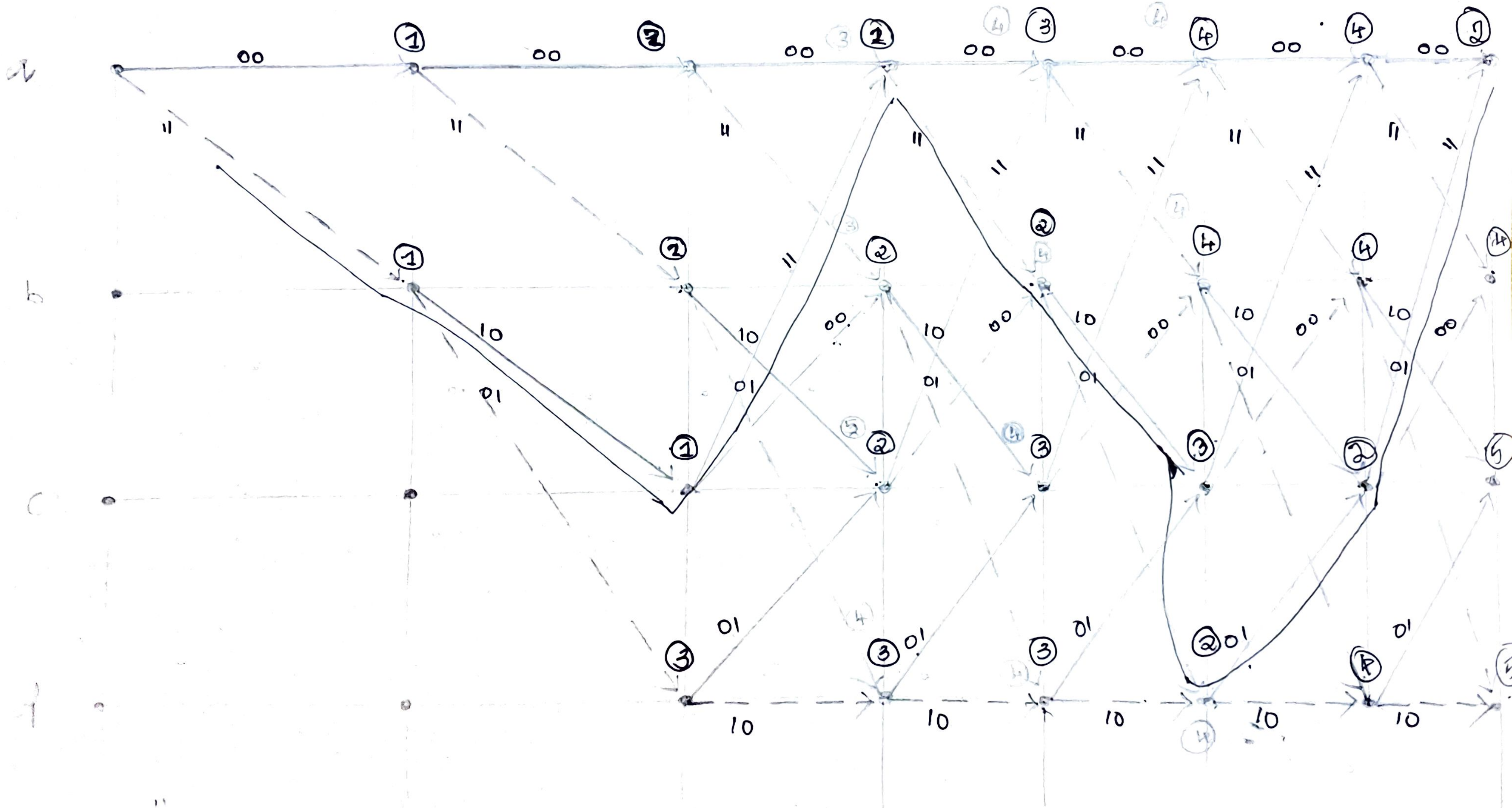
$g^{(2)} = (1\ 1\ 0\ 1)$



$x_1 = m \oplus m_3$

$x_2 = m \oplus m_1 \oplus m_3$

01 10 10 11 01 01 11

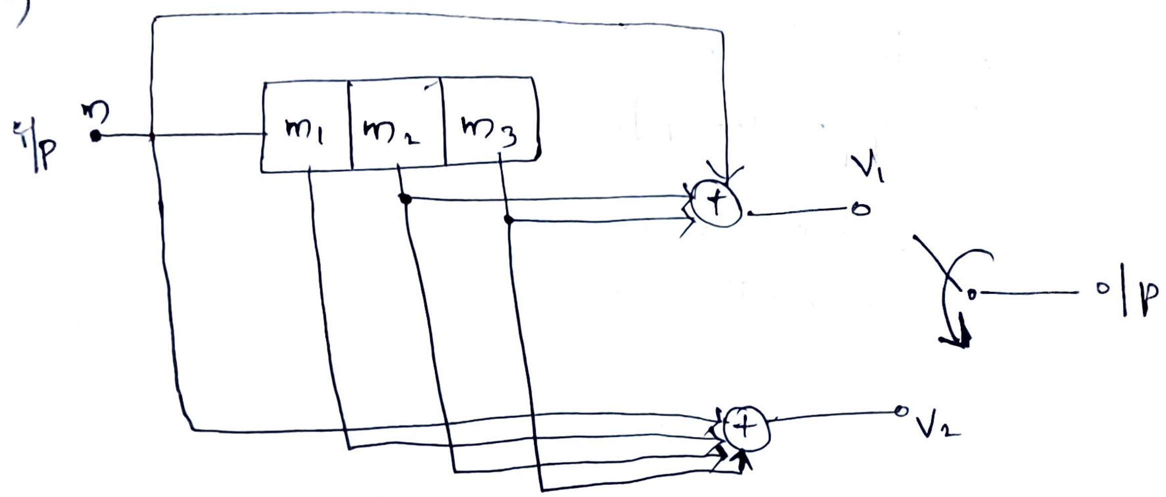


11 10 11 11 01 01 11

$(2, 1, 3)$ $\quad g^{(1)} = (1011)$

n, k, m $\quad g^{(2)} = (1111)$

i)



$V_1 = m \oplus m_2 \oplus m_3$

$V_2 = m \oplus m_1 \oplus m_2 \oplus m_3$

State	m	m_1	m_2	m_3	m	m_1	m_2	o/p
	0	0	0	0				
	1	0	0	0				
	0	0	0	1				
	1	0	0	1				
	0	0	1	0				
	1	0	1	0				
	0	0	1	1				
	1	0	1	1				
	0	1	0	0				
	1	1	0	0				
	0	1	0	1				

$e = mG$

length Code word $= n(L+m)$

$= 2(4+3)$

$= 2 \times 7$

$= \underline{\underline{14}}$

$g^{(1)} = (1011)$ $\quad g^{(2)} = (1111)$ $\quad u = \underline{\underline{1101}}$

$x_1 = u \circledast g^{(1)}$

$x_1 = 11111\ 11$

	1	1	0	1
1	1	1	0	1
0	0	0	0	0
1	1	1	0	1
1	1	1	0	1

$x_2 = u * g^{(2)}$

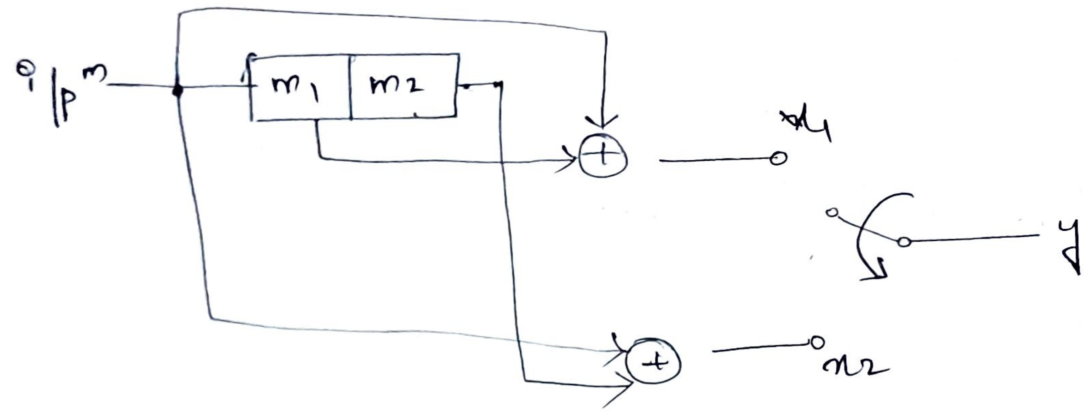
$= 1\ 0\ 0\ 1\ 0\ 1\ 1$

	1	1	0	1
1	1	1	0	1
1	1	1	0	1
1	1	1	0	1
1	1	1	0	1

o/p $y = x_1 x_2$

$y = \underline{11\ 10\ 10\ 11\ \ 10\ \ 11\ \ 11}$

$g^{(1)} = (1, 1, 0) \qquad g^{(2)} = (1, 0, 1)$



$g^{(1)}(x) = (1\ 1\ 0) = 1 + x$

$g^{(2)}(x) = (1, 0, 1) = 1 + x^2$

msg $M(x) = (1 0 1 1) = (1 + x^2 + x^3)$

$V_1(x) = g^{1}(x)\, m(x) = (1+x)(1+x^2+x^3)$

$= 1 + x^2 + \cancel{x^3} + x + \cancel{x^3} + x^4$

$= 1 + x + x^2 + x^4$

$V_1 = \underline{11101}$

$V_2(x) = g^{2}(x)\, m(x) = (1+x^2)(1+x^2+x^3)$

$= 1 + \cancel{x^2} + x^3 + \cancel{x^2} + x^4 + x^5$

$= 1 + x^3 + x^4 + x^5$

$V_2 = \underline{10011 1}$

To make V_1 and V_2's no. of bits equal

$V_1 = 111010$

$V_2 = 100111$

$y = 11\ 10\ 10\ 01\ 11\ 10$

$length(y) = n(L+m)$

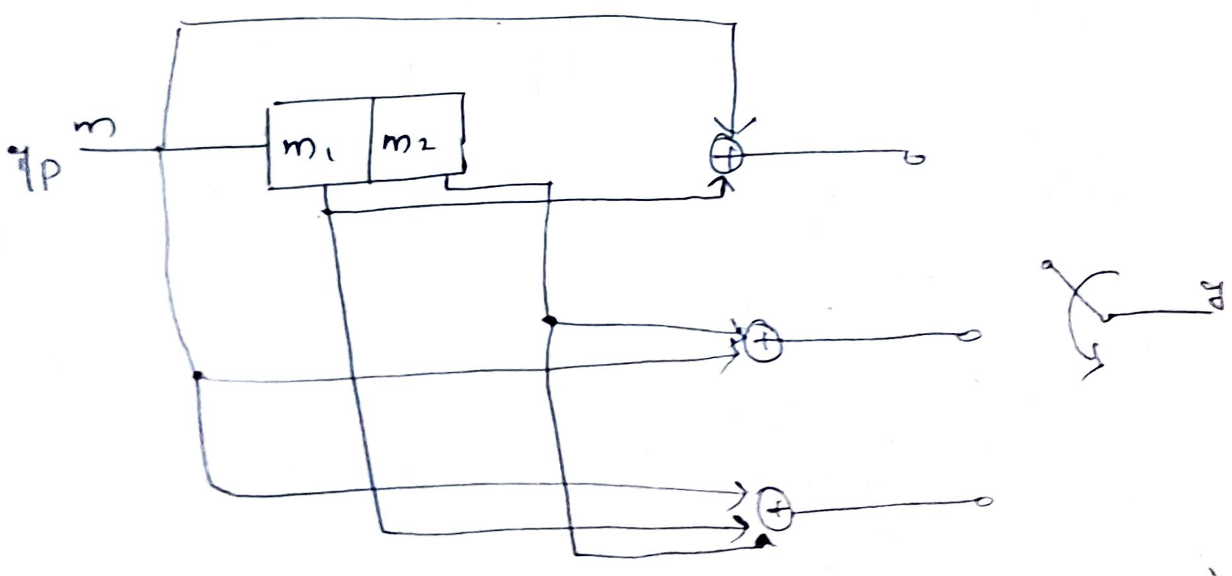
$= 2(4+2)$

$= 2(6)$

$= 12$

a) $(3, 1, 2)$ (n, k, m) $g^{(1)} = (1\ 1\ 0)$ $g^{(2)} = (1\ 0\ 1)$ $g^{(3)} = (1\ 1\ 1)$

i)



$L = length(11101)$

$= 5$

ii) $G = L \times (n(L+m))$

$= 5 \times (3(5+2))$

$5 \times (3 \times 7)$

$= \underbrace{5}_{row} \times \underbrace{21}_{column}$

$$G = \begin{bmatrix} 110 & 101 & 111 & 000 & 000 & 000 & 000 \\ 000 & 110 & 101 & 111 & 000 & 000 & 000 \\ 000 & 000 & 110 & 101 & 111 & 000 & 000 \\ 000 & 000 & 000 & 110 & 101 & 111 & 000 \\ 000 & 000 & 000 & 000 & 110 & 101 & 111 \end{bmatrix}$$

ii) Time domain approach.

length $(y) = n(L+m)$
$= 3(5+2)$
$= 3 \times 7 = \underline{\underline{21}}$

$y = u\,G$

$= [1\ 1\ 1\ 0\ 1][G]$

OR

$V_1 = u * g^{(1)}$

$= \underline{1\,0\,0\,1\,1\,1\,0}$

	1	1	1	0	1
1	1	1	1	0	1
1	1	1	1	0	1
0	0	0	0	0	0

$V_2 = u * g^{(2)}$

$= \underline{\underline{1\,1\,0\,1\,0\,0\,1}}$

	1	1	1	0	1
1	1	1	1	0	1
0	0	0	0	0	0
1	1	1	1	0	1

$V_3 = u * g^{(3)}$

$= \underline{\underline{1\,0\,1\,0\,0\,1\,1}}$

	1	1	1	0	1
1	1	1	1	0	1
1	1	1	1	0	1
1	1	1	1	0	1

$y = V_1 V_2 V_3$

$= \underline{\underline{111\ 010\ 001\ 110\ 100\ 101\ 011}}$

Transform domain method.

$$u(x) = 1+x+x^2+x^4$$

$$g^{(1)}(x) = 110 = 1+x$$
$$g^{(2)}(x) = 101 = 1+x^2$$
$$g^{(3)}(x) = 111 = 1+x+x^2$$

$$V_1(x) = g^{(1)}(x)\,u(x)$$
$$= (1+x)(1+x+x^2+x^4)$$
$$= 1+\cancel{x}+\cancel{x^2}+x^4+\cancel{x}+\cancel{x^2}+x^3+x^5$$
$$= 1+x^3+x^4+x^5 = \underline{100111} = V_1$$

$$\Rightarrow \underline{1001110}$$

$$V_2(x) = g^{(2)}(x)\,u(x)$$
$$= (1+x^2)(1+x+x^2+x^4)$$
$$= 1+x+\cancel{x^2}+\cancel{x^4}+\cancel{x^2}+x^3+\cancel{x^4}+x^6$$
$$= 1+x+x^3+x^6 = \underline{1101001} = V_2$$

$$V_3(x) = g^{(3)}(x)\,u(x)$$
$$= (1+x+x^2)(1+x+x^2+x^4)$$
$$= 1+\cancel{x}+\cancel{x^2}+\cancel{x^4}+\cancel{x}+\cancel{x^2}+\cancel{x^3}+x^5+x^2+\cancel{x^3}+\cancel{x^4}+x^6$$
$$= 1+x^2+x^5+x^6 = \underline{1010011} = V$$

$y = V_1\ V_2\ V_3$

$= \underline{111\ 010\ 001\ 110\ 100\ 101\ 011}$